

# CHARACTERISTIC VARIETIES AND BETTI NUMBERS OF FREE ABELIAN COVERS

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**ABSTRACT.** The regular  $\mathbb{Z}^r$ -covers of a finite cell complex  $X$  are parameterized by the Grassmannian of  $r$ -planes in  $H^1(X, \mathbb{Q})$ . Moving about this variety, and recording when the Betti numbers  $b_1, \dots, b_i$  of the corresponding covers are finite carves out certain subsets  $\Omega_r^i(X)$  of the Grassmannian.

We present here a method, essentially going back to Dwyer and Fried, for computing these sets in terms of the jump loci for homology with coefficients in rank 1 local systems on  $X$ . Using the exponential tangent cones to these jump loci, we show that each  $\Omega$ -invariant is contained in the complement of a union of Schubert varieties associated to an arrangement of linear subspaces in  $H^1(X, \mathbb{Q})$ .

The theory can be made very explicit in the case when the characteristic varieties of  $X$  are unions of translated tori. But even in this setting, the  $\Omega$ -invariants are not necessarily open, not even when  $X$  is a smooth complex projective variety. As an application, we discuss the geometric finiteness properties of some classes of groups.

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## 1. INTRODUCTION

In a short yet insightful paper [21], W. Dwyer and D. Fried showed that the support loci of the Alexander invariants of a finite CW-complex completely determine the (rational)

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homological finiteness properties of its regular, free abelian covers. This result was recast in [35] in terms of the characteristic varieties of the given CW-complex, as well as their exponential tangent cones.

Our goal here is to lay the foundations of this theory in some detail, develop the machinery to a fuller extent, present a number of new results and applications, especially in the setting of smooth, complex quasi-projective varieties, and indicate some further directions of study.

**1.1. Characteristic varieties.** The origins of the subject go back to the 1920s, when J.W. Alexander introduced his eponymous knot polynomial. Let  $K$  be a knot in  $S^3$ , let  $X = S^3 \setminus K$  be its complement, and let  $X^{\text{ab}} \rightarrow X$  be the universal abelian cover. Then, the first homology group  $H_1(X^{\text{ab}}, \mathbb{Z})$  is a finitely presented, torsion module over the group-ring  $\mathbb{Z}[H_1(X, \mathbb{Z})] \cong \mathbb{Z}[t^{\pm 1}]$ ; the Alexander polynomial of the knot,  $\Delta_K(t)$ , is simply the order of this module. More generally, if  $L$  is an  $n$ -component link in  $S^3$ , there is a multi-variable Alexander polynomial,  $\Delta_L(t_1, \dots, t_n)$ , depending only on the link group, and a choice of meridians.

Even more generally, let  $X$  be a connected CW-complex with finite  $k$ -skeleton, for some  $k \geq 1$ . Each homology group  $H_j(X^{\text{ab}}, \mathbb{C})$  with  $j \leq k$  is a finitely generated module over the Noetherian ring  $\mathbb{C}[G_{\text{ab}}]$ , where  $G = \pi_1(X, x_0)$ . Consider the support loci for the direct sum of these modules, up to a fixed degree  $i \leq k$ :

$$(1) \quad \mathcal{V}^i(X) = V\left(\text{ann}\left(\bigoplus_{j \leq i} H_j(X^{\text{ab}}, \mathbb{C})\right)\right).$$

By construction, these loci are Zariski closed subsets of the character group  $\widehat{G} = \text{Hom}(G, \mathbb{C}^\times)$ . As shown in [35], the varieties defined in this fashion may be reinterpreted as the jump loci for the homology of  $X$  with coefficients in rank 1 local systems:

$$(2) \quad \mathcal{V}^i(X) = \{\rho \in \widehat{G} \mid H_j(X, \mathbb{C}_\rho) \neq 0, \text{ for some } j \leq i\}.$$

It turns out that the geometry of these *characteristic varieties* is intimately related to the homological properties of regular, abelian covers of  $X$ . For the purpose of studying free abelian covers (like we do here), it will be enough to consider the subvarieties  $\mathcal{W}^i(X)$ , obtained by intersecting  $\mathcal{V}^i(X)$  with the identity component of the character group,  $\widehat{G}^\circ$ .

**1.2. The Dwyer–Fried sets.** For a fixed positive integer  $r$ , the connected, regular covers  $Y \rightarrow X$  with group of deck-transformations free abelian of rank  $r$  are parameterized by the Grassmannian of  $r$ -planes in the rational vector space  $H^1(X, \mathbb{Q})$ . Moving about this variety, and recording when all the Betti numbers  $b_1(Y), \dots, b_i(Y)$  are finite defines subsets

$$(3) \quad \Omega_r^i(X) \subseteq \text{Gr}_r(H^1(X, \mathbb{Q})),$$

which we call the *Dwyer–Fried invariants* of  $X$ . As with the characteristic varieties, these sets depend only on the homotopy type of the given CW-complex  $X$ . Consequently, if  $G$  is a finitely generated group, we may define  $\Omega_r^i(G) := \Omega_r^i(K(G, 1))$ .

In [21], Dwyer and Fried showed that the  $\Omega$ -sets (3) are completely determined by the support varieties (1). Building on this work, and on the refinements from [35], we recast this result in Theorem 4.3, as follows. Identify the character group  $\widehat{G}$  with  $H^1(X, \mathbb{C}^\times)$ , and let  $\exp: H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathbb{C}^\times)$  be the coefficient homomorphism induced by the exponential map  $\exp: \mathbb{C} \rightarrow \mathbb{C}^\times$ . Then,

$$(4) \quad \Omega_r^i(X) = \{P \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid \dim_{\mathbb{C}}(\exp(P \otimes \mathbb{C}) \cap \mathcal{W}^i(X)) = 0\}.$$

In other words, if  $P$  is an  $r$ -plane in  $H^1(X, \mathbb{Q})$  and  $Y \rightarrow X$  is the corresponding  $\mathbb{Z}^r$ -cover, then the first  $i$  Betti numbers of  $Y$  are finite if and only if the intersection of the algebraic torus  $\exp(P \otimes \mathbb{C})$  with the characteristic variety  $\mathcal{W}^i(X)$  is finite.

**1.3. An upper bound for the  $\Omega$ -sets.** Each characteristic variety  $\mathcal{W}^i(X)$  determines an arrangement  $\mathcal{C}_i(X)$  of rational subspaces in  $H^1(X, \mathbb{Q})$ . Indeed, let  $\tau_1(\mathcal{W}^i(X))$  be the set of points  $z \in H^1(X, \mathbb{C})$  such that  $\exp(\lambda z)$  belongs to  $\mathcal{W}^i(X)$ , for all  $\lambda \in \mathbb{C}$ . As shown in [20], and as reproved in detail here in Theorem 5.5, this set is a finite union of rationally defined subspaces. We then simply declare that  $\tau_1(\mathcal{W}^i(X)) = \bigcup_{L \in \mathcal{C}_i(X)} L \otimes \mathbb{C}$ .

Using this notion, we establish in Theorems 6.1 and 7.4 the following upper bound for the Dwyer–Fried sets of our space  $X$ :

$$(5) \quad \Omega_r^i(X) \subseteq \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \bigcup_{L \in \mathcal{C}_i(X)} \sigma_r(L),$$

where  $\sigma_r(L)$  is the variety of incident  $r$ -planes to  $L$ . Thus, each set  $\Omega_r^i(X)$  is contained in the complement of a Zariski closed subset of  $\text{Gr}_r(H^1(X, \mathbb{Q}))$ , namely, the union of the special Schubert varieties  $\sigma_r(L)$  corresponding to the subspaces  $L$  in  $\mathcal{C}_i(X)$ .

If  $r = 1$ , inclusion (5) holds as equality; thus, the sets  $\Omega_1^i(X)$  are Zariski open subsets of  $\text{Gr}_1(H^1(X, \mathbb{Q}))$ . For  $r > 1$ , though, the sets  $\Omega_r^i(X)$  need not be open, not even in the usual topology on the Grassmannian. This rather surprising fact was first noticed by Dwyer and Fried, who constructed in [21] a 3-dimensional cell complex for which  $\Omega_2^2(X)$  is a finite set. We take here a different approach, by analyzing the openness of the  $\Omega$ -sets in a particularly simple, yet still intriguing case.

**1.4. Translated tori.** This is the case when all positive-dimensional components of the characteristic variety  $\mathcal{W}^i(X)$  are translated subtori of the character torus  $\widehat{G}^\circ$ . Write each such component as  $\rho_\alpha T_\alpha$ , where  $T_\alpha = \exp(L_\alpha \otimes \mathbb{C})$ , for some linear subspace  $L_\alpha \subset H^1(X, \mathbb{Q})$  and some  $\rho_\alpha \in \widehat{G}^\circ$ . Using the intersection theory of translated subgroups in a complex algebraic torus developed in [29, 47], we prove in Theorem 8.4 that

$$(6) \quad \Omega_r^i(X) = \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \bigcup_{\alpha} \sigma_r(L_\alpha, \rho_\alpha),$$

where  $\sigma_r(L_\alpha, \rho_\alpha)$  is the set of rational  $r$ -planes  $P$  for which  $\rho_\alpha \in \exp((P + L_\alpha) \otimes \mathbb{C})$  and  $P \cap L_\alpha \neq \{0\}$ . These sets, which generalize the usual Schubert varieties  $\sigma_r(L_\alpha)$ , need not be Zariski closed.

One instance when this happens is described in Theorem 8.5. Suppose there is a component  $\rho_\beta T_\beta$  of dimension  $d \geq 2$  such that it, and all other components parallel to it have non-trivial, finite-order translation factors, while the other components  $\rho_\alpha T_\alpha$  satisfy  $\tau_1(T_\alpha) \cap \tau_1(T_\beta) = \{0\}$ . Then, for each  $2 \leq r \leq d$ , the set  $\Omega_r^i(X)$  is *not* an open subset of  $\text{Gr}_r(H^1(X, \mathbb{Q}))$ .

We illustrate this phenomenon in Example 8.7, where we construct a finitely presented group  $G$  for which  $\Omega_2^1(G)$  consists of a single point. As far as we know, this is the first example of a group for which one of the Dwyer–Fried sets is not open.

**1.5. The Green–Lazarsfeld sets and the  $\Omega$ -sets.** Perhaps the best-known class of spaces for which the characteristic varieties consist only of translated tori is that of compact Kähler manifolds. The basic structure of the characteristic varieties of these manifolds was determined by Green and Lazarsfeld [25, 26], building on work of Beauville [6] and Catanese [13]. The theory was further amplified in [39, 22, 3, 11], with some of the latest developments appearing in [17, 16, 20, 10, 12, 4].

In particular, suppose  $M$  is a compact Kähler manifold, and  $T$  is a positive-dimensional component of  $\mathcal{V}^1(M)$ . There is then an orbifold fibration  $f: M \rightarrow \Sigma_{g,\mathbf{m}}$  with either base genus  $g \geq 2$ , or with  $g = 1$  and non-trivial multiplicity vector  $\mathbf{m}$ , so that  $T$  is a component of the pullback along  $f_\sharp: \pi_1(M) \rightarrow \Gamma$  of the first characteristic variety of the orbifold fundamental group  $\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,\mathbf{m}})$ . Now, it is readily seen that  $\mathcal{V}^1(\Gamma) = \widehat{\Gamma}$  or  $\widehat{\Gamma} \setminus \widehat{\Gamma}^\circ$ , depending on whether the base genus is at least 2 or not, and this finishes the description of the positive-dimensional components of  $\mathcal{V}^1(M)$ .

This description allows us to either estimate or compute explicitly the degree one Dwyer–Fried sets of  $M$ . For instance, we show in Theorem 9.10 that

$$(7) \quad \Omega_r^1(M) \subseteq \text{Gr}_r(H^1(M, \mathbb{Q})) \setminus \bigcup_{\alpha} \sigma_r(f_\alpha^*(H^1(\Sigma_{g_\alpha}, \mathbb{Q}))),$$

where the union runs over the set of orbifold fibrations  $f_\alpha: M \rightarrow \Sigma_{g_\alpha}$  with  $g_\alpha \geq 2$ . Moreover, if  $r = 1$ , or if there are no orbifold fibrations with multiple fibers, then (7) holds as equality.

In general, though, the above inclusion is strict. For instance, suppose  $M$  is a smooth, complex projective variety, admitting an elliptic pencil with multiple fibers. We then show in Proposition 9.12 that  $\Omega_2^1(M)$  is *not* an open subset of the Grassmannian. A concrete example of this phenomenon is provided by the Catanese–Ciliberto–Mendes Lopes surface (the quotient of  $\Sigma_2 \times \Sigma_3$  by a certain free involution), which fibers over  $\Sigma_1$  with two multiple fibers, each of multiplicity 2.

**1.6. Quasi-projective varieties and hyperplane arrangements.** Much of this theory generalizes to quasi-Kähler manifolds. Indeed, if  $X = \overline{X} \setminus D$  is such a manifold, obtained from a compact Kähler manifold  $\overline{X}$  with  $b_1(\overline{X}) = 0$  by removing a normal-crossings divisor  $D$ , then a theorem of Arapura [3] insures that all components of  $\mathcal{V}^i(X)$  are unitary translates of algebraic subtori.

Now, if  $X$  is a smooth, quasi-projective variety, then all such subtori in  $\mathcal{V}^1(X)$  arise by pullback along certain orbifold fibrations  $f: X \rightarrow (\Sigma_{g,s}, \mathbf{m})$  to Riemann surfaces of genus  $g \geq 0$  with  $s \geq 0$  points removed. Using this fact, we show in Proposition 10.10 that

$$(8) \quad \Omega_r^1(X) \subseteq \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \bigcup_{\alpha} \sigma_r(f_\alpha^*(H^1(\Sigma_{g_\alpha, s_\alpha}, \mathbb{Q}))),$$

where the union runs over the set of orbifold fibrations  $f_\alpha: X \rightarrow \Sigma_{g_\alpha, s_\alpha}$  with  $2g_\alpha + s_\alpha \geq 3$ . As before, if  $r = 1$ , or if there are no orbifold fibrations with multiple fibers, then (8) holds as equality. On the other hand, we prove in Corollary 10.13 the following: If there is a pencil  $X \rightarrow \mathbb{C}^\times$  with multiple fibers, but there is no pencil  $X \rightarrow \Sigma_{g,s}$  with  $b_1(X) - b_1(\Sigma_{g,s}) \leq 1$ , then inclusion (8) is strict for  $r = 2$ .

In the case of hyperplane arrangements, the whole theory becomes much more specific and combinatorial in nature. If  $\mathcal{A}$  is an arrangement of  $n$  hyperplanes in  $\mathbb{C}^d$ , then its complement,  $X(\mathcal{A})$ , is a smooth, quasi-projective variety. Its first characteristic variety,  $\mathcal{V}^1(X(\mathcal{A}))$ , is a union of torsion-translated subtori in  $(\mathbb{C}^\times)^n$ . As shown by Falk and Yuzvinsky in [24], the components passing through the origin correspond to multinets on the intersection lattice of  $\mathcal{A}$ . The other components are either torsion-translates of the former, isolated torsion points, or 1-dimensional translated subtori arising from orbifold fibrations  $X(\mathcal{A}) \rightarrow \mathbb{C}^\times$  with at least one multiple fiber.

This description of the first characteristic variety of an arrangement leads to the following combinatorial upper bound for the degree one Dwyer–Fried sets:

$$(9) \quad \Omega_r^1(X(\mathcal{A})) \subseteq \text{Gr}_r(\mathbb{Q}^n) \setminus \bigcup_{\alpha} \sigma_r(L_\alpha),$$

where the union runs over the set of orbifold fibrations  $f_\alpha: X(\mathcal{A}) \rightarrow \Sigma_{0,s_\alpha}$ ,  $s_\alpha \geq 3$ , corresponding to multinets on  $L(\mathcal{A})$ , and  $L_\alpha = f_\alpha^*(H^1(\Sigma_{0,s_\alpha}, \mathbb{Q}))$ . For certain classes of arrangements (e.g., line arrangements for which one or two lines contain all the intersection points of multiplicity 3 and higher), equality holds in (9). In the case of the deleted  $B_3$  arrangement, though, the presence of a 1-dimensional translated subtorus in the first characteristic variety forces a strict inclusion for  $r = 2$ .

**1.7. Geometric finiteness.** To conclude, we investigate the relationship between the Dwyer–Fried invariants and other, well-known finiteness properties of groups, namely, Wall’s property  $F_k$  and Serre and Bieri’s property  $FP_k$ . Using an idea that goes back to J. Stallings’ pioneering work on the subject, [40], we show that the  $\Omega$ -sets provide some useful information about those properties, too.

For instance, let  $\Gamma$  be a group of type  $F_{k-1}$ , i.e., a group admitting a classifying space with finite  $(k-1)$ -skeleton, and suppose  $\Gamma$  arises as the kernel of an epimorphism  $G \rightarrow \mathbb{Z}^r$ , where  $G$  is a finitely generated group with  $\Omega_r^k(G) = \emptyset$ . We then show in Theorem 12.1 that  $H_k(\Gamma, \mathbb{Z})$  is not finitely generated; in particular,  $\Gamma$  is not of type  $FP_k$ .

Motivated by the long-standing Shafarevich conjecture on holomorphic convexity, J. Kollár asked in [30] the following question: Is the fundamental group of a smooth, complex projective variety commensurable (up to finite kernels) with a group admitting a smooth, quasi-projective variety as classifying space? In [19], we gave a negative answer to this question, by proving the following result (recorded here as Theorem 12.5): For each  $k \geq 3$ , there is a smooth, complex projective variety  $M$  of complex dimension  $k-1$ , such that the group  $\Gamma = \pi_1(M)$  is of type  $F_{k-1}$ , but not of type  $FP_k$ . Using the machinery developed here—in particular, Theorem 12.1—we end the paper with a shorter and more transparent proof of the main result from [19].

**1.8. Further directions.** In a companion paper [45], we connect the Dwyer–Fried invariants of a space  $X$  to the resonance varieties of the cohomology algebra  $H^*(X, \mathbb{C})$ , under certain “straightness” assumptions. More precisely, suppose that, for each  $i \leq k$ , all components of  $\mathcal{W}^i(X)$  passing through the origin of  $H^1(X, \mathbb{C}^\times)$  are algebraic subtori, and the tangent cone at 1 to  $\mathcal{W}^i(X)$  equals the  $i$ th resonance variety,

$$(10) \quad \mathcal{R}^i(X, \mathbb{C}) = \{a \in H^1(X, \mathbb{C}) \mid H^j(H^*(X, \mathbb{C}), \cdot a) \neq 0, \text{ for some } j \leq i\}.$$

Then, for all  $i \leq k$  and  $r \geq 1$ , the set  $\Omega_r^i(X) \subseteq \text{Gr}_r(H^1(X, \mathbb{Q}))$  is contained in the complement to the incidence variety  $\sigma_r(\mathcal{R}^i(X, \mathbb{Q}))$ . If, moreover, all positive-dimensional components of  $\mathcal{W}^i(X)$  pass through the origin, then this inclusion holds as equality.

In joint work with Y. Yang and G. Zhao, [48], we generalize the theory presented here to regular covers  $Y \rightarrow X$  for which the group of deck-transformations is a fixed, finitely generated (not necessarily torsion-free) abelian group  $A$ . Such covers are parametrized by epimorphisms from  $\pi_1(X)$  to  $A$ , modulo the action of the automorphism group of  $A$  on the target. Inside this parameter space, we single out the subsets  $\Omega_A^i(X)$  consisting of all  $A$ -covers with finite Betti numbers up to degree  $i$ . These sets can again be computed in terms of intersections of algebraic subtori with the characteristic varieties. For many spaces of interest, the homological finiteness of abelian covers can be tested through the corresponding free abelian covers; arbitrary abelian covers, though, exhibit different homological finiteness properties than their free abelian counterparts.

Finally, in [46] we explore some of the connections between the Dwyer–Fried invariants  $\Omega_r^i(X) \subseteq \text{Gr}_r(H^1(X, \mathbb{Q}))$  and the Bieri–Neumann–Strebel–Renz invariants  $\Sigma^i(X, \mathbb{Z}) \subseteq$

$H^1(X, \mathbb{R})$ . In [35], we showed that

$$(11) \quad \Sigma^i(X, \mathbb{Z}) \subseteq H^1(X, \mathbb{R}) \setminus \bigcup_{L \in \mathcal{C}_i(X)} L \otimes \mathbb{R}.$$

This (sharp) upper bound for the  $\Sigma$ -invariants may be viewed as the analogue of the upper bound (5) for the  $\Omega$ -invariants. Building on the work presented here, we prove in [46] the following theorem: If equality holds in (11), then equality must also hold in (5). This result allows us to derive useful information about the notoriously intricate  $\Sigma$ -invariants from concrete knowledge of the more accessible  $\Omega$ -invariants. For instance, a basic question about the nature of the  $\Sigma$ -invariants of arrangements was posed at an Oberwolfach Mini-Workshop in 2007, and was later revisited in [35, 44]. Drawing on Example 11.8, we provide in [46] an answer to this question, thereby laying the ground for further study of the homological and geometric finiteness properties of arrangements.

## 2. EQUIVARIANT CHAIN COMPLEX AND CHARACTERISTIC VARIETIES

We start by reviewing the definition and basic properties of the characteristic varieties attached to a space.

**2.1. Homology in rank 1 local systems.** Let  $X$  be a connected CW-complex, with finite 1-skeleton. Without loss of generality, we may assume our space has a single 0-cell, call it  $x_0$ . Let  $G = \pi_1(X, x_0)$  be the fundamental group of  $X$ , based at  $x_0$ . Clearly, the group  $G$  is finitely generated (by the homotopy classes of the 1-cells of  $X$ ).

Denote by  $\mathbb{C}^\times$  the multiplicative group of non-zero complex numbers. The set of complex-valued characters,  $\widehat{G} = \text{Hom}(G, \mathbb{C}^\times)$ , is a commutative, affine algebraic group, with multiplication  $\rho_1 \cdot \rho_2(g) = \rho_1(g)\rho_2(g)$ , and identity the homomorphism taking constant value  $1 \in \mathbb{C}^\times$ .

Let  $G_{\text{ab}} = G/G' = H_1(X, \mathbb{Z})$  be the maximal abelian quotient of  $G$ . Since the group  $\mathbb{C}^\times$  is commutative, every character  $\rho: G \rightarrow \mathbb{C}^\times$  factors through  $G_{\text{ab}}$ . Thus, the abelianization map  $\text{ab}: G \rightarrow G_{\text{ab}}$  induces an isomorphism  $\widehat{\text{ab}}: \widehat{G_{\text{ab}}} \xrightarrow{\sim} \widehat{G}$ , which allows us to identify the coordinate ring of  $\widehat{G}$  with the group algebra  $\mathbb{C}[G_{\text{ab}}]$ . Write  $G_{\text{ab}} = \mathbb{Z}^n \oplus A$ , where  $n = b_1(G)$  and  $A$  is a finite abelian group. The identity component of the character group,  $\widehat{G}^\circ$ , is isomorphic to the complex algebraic torus  $(\mathbb{C}^\times)^n$ , while  $\widehat{G}$  is isomorphic to the product  $(\mathbb{C}^\times)^n \times \widehat{A}$ , where  $\widehat{A} \cong A$  is a subgroup of  $\mathbb{C}^\times$ , consisting of roots of unity. In particular, all components of  $\widehat{G}$  are  $n$ -dimensional algebraic tori.

The set  $\widehat{G}$  parametrizes rank 1 local systems on  $X$ : given a character  $\rho$ , denote by  $\mathbb{C}_\rho$  the complex vector space  $\mathbb{C}$ , viewed as a right module over the group ring  $\mathbb{Z}G$  via  $a \cdot g = \rho(g)a$ , for  $g \in G$  and  $a \in \mathbb{C}$ .

Let  $p: \widetilde{X} \rightarrow X$  be the universal cover. The cell structure on  $X$  lifts in a natural fashion to a cell structure on  $\widetilde{X}$ . Fixing a lift  $\tilde{x}_0 \in p^{-1}(x_0)$  identifies the group  $G = \pi_1(X, x_0)$  with the group of deck transformations of  $\widetilde{X}$ . Therefore, we may view the cellular chain complex of  $\widetilde{X}$ ,

$$(12) \quad \cdots \longrightarrow C_{i+1}(\widetilde{X}, \mathbb{Z}) \xrightarrow{\tilde{\partial}_{i+1}} C_i(\widetilde{X}, \mathbb{Z}) \xrightarrow{\tilde{\partial}_i} C_{i-1}(\widetilde{X}, \mathbb{Z}) \longrightarrow \cdots,$$

as a chain complex of left  $\mathbb{Z}G$ -modules. The homology of  $X$  with coefficients in  $\mathbb{C}_\rho$  is defined as the homology of the chain complex  $\mathbb{C}_\rho \otimes_{\mathbb{Z}G} C_\bullet(\widetilde{X}, \mathbb{Z})$ . In concrete terms,

$H_*(X, \mathbb{C}_\rho)$  may be computed from the chain complex of  $\mathbb{C}$ -vector spaces,

$$(13) \quad \cdots \longrightarrow C_{i+1}(X, \mathbb{C}) \xrightarrow{\tilde{\partial}_{i+1}(\rho)} C_i(X, \mathbb{C}) \xrightarrow{\tilde{\partial}_i(\rho)} C_{i-1}(X, \mathbb{C}) \longrightarrow \cdots,$$

where the evaluation of  $\tilde{\partial}_i$  at  $\rho$  is obtained by applying the ring homomorphism  $\mathbb{Z}G \rightarrow \mathbb{C}$ ,  $g \mapsto \rho(g)$  to each entry of  $\tilde{\partial}_i$ .

Alternatively, we may consider the universal abelian cover,  $X^{\text{ab}}$ , and its equivariant chain complex,  $C_*(X^{\text{ab}}, \mathbb{Z}) = \mathbb{Z}G_{\text{ab}} \otimes_{\mathbb{Z}G} C_*(\tilde{X}, \mathbb{Z})$ , with differentials  $\partial_i^{\text{ab}} = \text{id} \otimes \tilde{\partial}_i$ . The homology of  $X$  with coefficients in the rank 1 local system given by a character  $\rho \in \hat{G}_{\text{ab}} = \hat{G}$  is then computed from the chain complex (13), with differentials  $\partial_i^{\text{ab}}(\rho) = \tilde{\partial}_i(\rho)$ .

We will write  $b_i(X, \rho) = \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho)$ . Evidently, the identity  $1 \in \hat{G}$  yields the trivial local system,  $\mathbb{C}_1 = \mathbb{C}$ ; thus,  $H_*(X, \mathbb{C})$  is the usual homology of  $X$  with coefficients in  $\mathbb{C}$ , and  $b_i(X) = b_1(X, 1)$  is the  $i$ th Betti number of  $X$ .

**2.2. Jump loci for twisted homology.** Computing the homology groups of  $X$  with coefficients in rank 1 local systems carves out a notable collection of subsets of the character group  $\hat{G}$ .

**Definition 2.1.** Let  $X$  be a connected CW-complex with finite  $k$ -skeleton, for some  $k \geq 1$ . The *characteristic varieties* of  $X$  are the sets

$$\mathcal{V}_d^i(X) = \{\rho \in \hat{G} \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq d\},$$

defined for all degrees  $0 \leq i \leq k$  and all depths  $d > 0$ .

**Remark 2.2.** For the purpose of computing the characteristic varieties up to degree  $i = k$ , we may assume without loss of generality that  $X$  is a finite CW-complex of dimension  $k + 1$  (see [35, Lemma 2.1]).

The terminology from Definition 2.1, due to Libgober, is justified by the following lemma. For concreteness, we sketch a proof, based on an argument of Green and Lazarsfeld [25] (see also [31]). Given a commutative ring  $R$ , and a matrix  $\varphi: R^b \rightarrow R^a$ , denote by  $E_q(\varphi)$  the ideal generated by the minors of size  $a - q$  of  $\varphi$ .

**Lemma 2.3.** *The jump loci  $\mathcal{V}_d^i(X)$  are Zariski closed subsets of the algebraic group  $\hat{G}$ .*

*Proof.* Let  $R = \mathbb{C}[G_{\text{ab}}]$  be the coordinate ring of  $\hat{G} = \hat{G}_{\text{ab}}$ . By definition, a character  $\rho$  belongs to  $\mathcal{V}_d^i(X)$  if and only if  $\text{rank } \partial_{i+1}^{\text{ab}}(\rho) + \text{rank } \partial_i^{\text{ab}}(\rho) \leq c_i - d$ , where  $c_i = c_i(X)$  is the number of  $i$ -cells of  $X$ . Using this description, we may rewrite our jump locus as the zero-set of a sum of products of determinantal ideals,

$$(14) \quad \mathcal{V}_d^i(X) = V\left(\sum E_p(\partial_i^{\text{ab}}) \cdot E_q(\partial_{i+1}^{\text{ab}})\right),$$

with the sum running over all  $p, q \geq 0$  with  $p + q = c_{i-1} + d - 1$ .  $\square$

**2.3. Homotopy invariance.** Let  $X$  and  $Y$  be connected CW-complexes with finite  $k$ -skeleta, and denote by  $G$  and  $H$  the respective fundamental groups.

**Lemma 2.4.** *Suppose  $X \simeq Y$ . There is then an isomorphism  $\hat{H} \cong \hat{G}$ , which restricts to isomorphisms  $\mathcal{V}_d^i(Y) \cong \mathcal{V}_d^i(X)$ , for all  $i \leq k$ , and all  $d > 0$ .*

*Proof.* Let  $f: X \rightarrow Y$  be a homotopy equivalence; without loss of generality, we may assume  $f$  preserves skeleta. The induced homomorphism on fundamental groups,  $f_\# : G \rightarrow H$ , yields an isomorphism of algebraic groups,  $\hat{f}_\# : \hat{H} \rightarrow \hat{G}$ . Lifting  $f$  to a cellular homotopy equivalence,  $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ , defines isomorphisms  $H_i(X, \mathbb{C}_{\rho \circ f_\#}) \rightarrow H_i(Y, \mathbb{C}_\rho)$ , for



each character  $\rho \in \widehat{H}$ . Hence,  $\hat{f}_\#$  restricts to isomorphisms  $\mathcal{V}_d^i(Y) \rightarrow \mathcal{V}_d^i(X)$  between the respective characteristic varieties.  $\square$

In each fixed degree  $i$ , the characteristic varieties of a space  $X$  define a descending filtration on the character group,

$$(15) \quad \widehat{G} = \mathcal{V}_0^i(X) \supseteq \mathcal{V}_1^i(X) \supseteq \mathcal{V}_2^i(X) \supseteq \cdots.$$

Clearly,  $1 \in \mathcal{V}_d^i(X)$  if and only if  $d \leq b_i(X)$ . Moreover, if  $c_i(X) = 0$ , then  $\mathcal{V}_d^i(X) = \emptyset$ , for all  $d$ . In particular, if  $X$  has dimension  $k$ , then  $\mathcal{V}_d^i(X) = \emptyset$ , for all  $i > k$ , and all  $d > 0$ .

In degree 0, we have  $\mathcal{V}_1^0(X) = \{1\}$  and  $\mathcal{V}_d^0(X) = \emptyset$ , for  $d > 1$ . In degree 1, the characteristic varieties of  $X$  depend only on the fundamental group  $G = \pi_1(X, x_0)$ —in fact, only on its maximal metabelian quotient,  $G/G''$  (see §2.5 below). Accordingly, we will sometimes write  $\mathcal{V}_d^1(G)$  for  $\mathcal{V}_d^1(X)$ .

One may define in the same fashion the characteristic varieties  $\mathcal{V}_d^i(X, \mathbb{k})$  over an arbitrary field  $\mathbb{k}$ . Let us just note here that  $\mathcal{V}_d^i(X, \mathbb{k}) = \mathcal{V}_d^i(X, \mathbb{K}) \cap \text{Hom}(G, \mathbb{k}^\times)$ , for every field extension  $\mathbb{k} \subseteq \mathbb{K}$ .

**2.4. Some basic examples.** There are a few known instances where all the characteristic varieties of a given space can be computed explicitly. We record below some of these computations, which will be needed later on.

**Example 2.5.** We start with the circle  $S^1$ . Identify  $\pi_1(S^1, *) = \mathbb{Z}$ , and fix a generator  $t$ ; the cellular chain complex of  $\widetilde{S^1}$  then takes the form  $C_1 \xrightarrow{\partial_1} C_0$ , where  $C_1 = C_0 = \mathbb{Z}\mathbb{Z}$  and  $\partial_1(1) = t - 1$ . By specializing at a character  $\rho \in \widehat{\mathbb{Z}} = \mathbb{C}^\times$ , we get a chain complex of  $\mathbb{C}$ -vector spaces,  $\mathbb{C} \xrightarrow{\rho-1} \mathbb{C}$ , which is acyclic, except for  $\rho = 1$ , when  $H_0(S^1, \mathbb{C}) = H_1(S^1, \mathbb{C}) = \mathbb{C}$ . Hence,  $\mathcal{V}_1^0(S^1) = \mathcal{V}_1^1(S^1) = \{1\}$  and  $\mathcal{V}_d^i(S^1) = \emptyset$ , otherwise.

**Example 2.6.** Let  $\bigvee^n S^1$  be a wedge of  $n$  circles,  $n > 1$ . Then  $\pi_1(\bigvee^n S^1) = F_n$ , the free group of rank  $n$ . It is readily seen that

$$(16) \quad \mathcal{V}_d^i\left(\bigvee^n S^1\right) = \begin{cases} (\mathbb{C}^\times)^n & \text{if } i = 1 \text{ and } d < n, \\ \{1\} & \text{if } i = 1 \text{ and } d = n, \text{ or } i = 0 \text{ and } d = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

**Example 2.7.** Let  $\Sigma_g$  be a compact, connected, orientable surface of genus  $g \geq 1$ , and identify  $\text{Hom}(\pi_1(\Sigma_g), \mathbb{C}^\times) = (\mathbb{C}^\times)^{2g}$ . We then have

$$(17) \quad \mathcal{V}_d^i(\Sigma_g) = \begin{cases} (\mathbb{C}^\times)^{2g} & \text{if } i = 1 \text{ and } d < 2g - 1, \\ \{1\} & \text{if } i = 1 \text{ and } d \in \{2g - 1, 2g\}, \text{ or } i \in \{0, 2\} \text{ and } d = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

**2.5. Depth one characteristic varieties.** Most important for us will be the depth-1 characteristic varieties,  $\mathcal{V}_1^i(X)$ , and their unions up to a fixed degree,  $\mathcal{V}^i(X) = \bigcup_{j=0}^i \mathcal{V}_j^i(X)$ . Let  $G = \pi_1(X, x_0)$ . Clearly,

$$(18) \quad \mathcal{V}^i(X) = \{\rho \in \widehat{G} \mid b_j(X, \rho) \neq 0, \text{ for some } j \leq i\},$$

These varieties yield an ascending filtration of the character group,  $\{1\} = \mathcal{V}^0(X) \subseteq \mathcal{V}^1(X) \subseteq \cdots \subseteq \mathcal{V}^k(X) \subseteq \widehat{G}$ .

It follows from [35, Corollary 3.7], that the sets (18) are the support varieties for the Alexander invariants of  $X$ . More precisely, view  $H_*(X^{\text{ab}}, \mathbb{C})$  as a module over the



group-ring  $\mathbb{C}G_{\text{ab}}$ . Then,

$$(19) \quad \mathcal{V}^i(X) = V\left(\text{ann}\left(\bigoplus_{j \leq i} H_j(X^{\text{ab}}, \mathbb{C})\right)\right).$$

In particular,  $\mathcal{V}^1(G) = V(\text{ann}(H_1(G', \mathbb{Z}) \otimes \mathbb{C}))$ , where the  $\mathbb{Z}G_{\text{ab}}$ -module structure on the group  $H_1(G', \mathbb{Z}) = G'/G''$  arises from the extension  $0 \rightarrow G'/G'' \rightarrow G/G'' \rightarrow G'/G'' \rightarrow 0$ . This shows that the characteristic variety  $\mathcal{V}^1(G)$  does indeed depend only on the maximal metabelian quotient  $G/G''$ .

We will also consider the varieties  $\mathcal{W}^i(X) = \mathcal{V}^i(X) \cap \widehat{G}^\circ$  inside the complex algebraic torus  $\widehat{G}^\circ$ . An alternate description of these varieties is as follows. Let  $X^\alpha \rightarrow X$  be the maximal torsion-free abelian cover of  $X$ , corresponding to the projection  $\alpha: G \rightarrow G_{\text{ab}}/\text{Tors}(G_{\text{ab}}) = \mathbb{Z}^n$ , where  $n = b_1(G)$ . Identify  $\widehat{G}^\circ = (\mathbb{C}^\times)^n$ , and the group ring  $\mathbb{C}\mathbb{Z}^n$  with the Laurent polynomial ring  $\Lambda_n = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ . Then,

$$(20) \quad \mathcal{W}^i(X) = V\left(\text{ann}\left(\bigoplus_{j \leq i} H_j(X^\alpha, \mathbb{C})\right)\right).$$

If  $X$  has finite 2-skeleton, the set  $\mathcal{W}^1(X)$  can be computed from a finite presentation for the fundamental group, by means of the Fox free differential calculus. Suppose  $G = \langle x_1, \dots, x_q \mid r_1, \dots, r_m \rangle$ , and let  $\Phi_G$  be the corresponding  $m$  by  $q$  Alexander matrix, with entries  $\alpha(\partial r_i / \partial x_j)$  obtained by applying the ring morphism  $\alpha: \mathbb{Z}G \rightarrow \Lambda_n$  to the Fox derivatives of the relators. The variety  $\mathcal{W}^1(G) = \mathcal{W}^1(X)$ , then, is defined by the vanishing of the codimension 1 minors of  $\Phi_G$ , at least away from the trivial character 1.

**Example 2.8.** Let  $L = (L_1, \dots, L_n)$  be a link of smoothly embedded circles in  $S^3$ , with complement  $X = S^3 \setminus \bigcup_{i=1}^n L_i$ . Choosing orientations on the link components yields a preferred basis for  $H_1(X, \mathbb{Z}) = \mathbb{Z}^n$  consisting of oriented meridians. Using this basis, identify  $H^1(X, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$ . Then

$$(21) \quad \mathcal{W}^1(X) = \{\zeta \in (\mathbb{C}^\times)^n \mid \Delta_L(\zeta) = 0\} \cup \{1\},$$

where  $\Delta_L = \Delta_L(t_1, \dots, t_n)$  is the (multi-variable) Alexander polynomial of the link. For details and references on this, see [44].

**2.6. Products and wedges.** The depth-1 characteristic varieties behave well with respect to products and wedges. To make this statement precise, let  $X_1$  and  $X_2$  be connected CW-complexes with finite  $k$ -skeleta, and with fundamental groups  $G_1$  and  $G_2$ .

We start with the product  $X = X_1 \times X_2$ . Identify  $G = \pi_1(X)$  with  $G_1 \times G_2$ ; also,  $\widehat{G} = \widehat{G}_1 \times \widehat{G}_2$  and  $\widehat{G}^\circ = \widehat{G}_1^\circ \times \widehat{G}_2^\circ$ .

**Proposition 2.9** ([35]).  $\mathcal{V}_1^i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{V}_1^p(X_1) \times \mathcal{V}_1^q(X_2)$ , for all  $i \leq k$ .

The idea of the proof is very simple. For each character  $\rho = (\rho_1, \rho_2) \in \widehat{G}$ , the chain complex  $C_\bullet(X, \mathbb{C}_\rho)$  decomposes as  $C_\bullet(X_1, \mathbb{C}_{\rho_1}) \otimes_{\mathbb{C}} C_\bullet(X_2, \mathbb{C}_{\rho_2})$ . Taking homology, we get  $H_i(X, \mathbb{C}_\rho) = \bigoplus_{s+t=i} H_s(X_1, \mathbb{C}_{\rho_1}) \otimes_{\mathbb{C}} H_t(X_2, \mathbb{C}_{\rho_2})$ , and the claim follows.

**Corollary 2.10.** For all  $i \leq k$ , we have that  $\mathcal{V}^i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{V}^p(X_1) \times \mathcal{V}^q(X_2)$  and  $\mathcal{W}^i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{W}^p(X_1) \times \mathcal{W}^q(X_2)$ .

Now consider the wedge,  $X = X_1 \vee X_2$ , taken at the unique 0-cells. Identify  $G = \pi_1(X)$  with  $G_1 * G_2$ ; also,  $\widehat{G} = \widehat{G}_1 \times \widehat{G}_2$  and  $\widehat{G}^\circ = \widehat{G}_1^\circ \times \widehat{G}_2^\circ$ .

**Proposition 2.11** ([35]). *Suppose  $X_1$  and  $X_2$  have positive first Betti numbers. Then, for all  $1 \leq i \leq k$ ,*

$$\mathcal{V}_1^i(X_1 \vee X_2) = \begin{cases} \widehat{G}_1 \times \widehat{G}_2 & \text{if } i = 1, \\ \mathcal{V}_1^i(X_1) \times \widehat{G}_2 \cup \widehat{G}_1 \times \mathcal{V}_1^i(X_2) & \text{if } i > 1. \end{cases}$$

The proof uses the decomposition  $C_+(X, \mathbb{C}_\rho) = C_+(X_1, \mathbb{C}_{\rho_1}) \oplus C_+(X_2, \mathbb{C}_{\rho_2})$ . Taking homology, we get  $b_i(X, \rho) = b_i(X_1, \rho_1) + b_i(X_2, \rho_2) + \epsilon$ , where  $\epsilon = 1$  if  $i = 1$ ,  $\rho_1 \neq 1$ , and  $\rho_2 \neq 1$ , and  $\epsilon = 0$ , otherwise. The claim follows.

**Corollary 2.12.** *If  $X_1$  and  $X_2$  have positive first Betti numbers, then  $\mathcal{V}^i(X_1 \vee X_2) = \widehat{G}$  and  $\mathcal{W}^i(X_1 \vee X_2) = \widehat{G}^\circ$ , for all  $i \leq k$ .*

The condition that  $b_1(X_1)$  and  $b_1(X_2)$  be positive may be dropped if  $i > 1$ , but not if  $i = 1$ . For instance, take  $X_1 = S^1$  and  $X_2 = S^2$ . Then  $G_1 = \mathbb{Z}$  and  $G_2 = \{1\}$ ; thus,  $\widehat{G} = \mathbb{C}^\times$ , yet  $\mathcal{V}^1(S^1 \vee S^2) = \{1\}$ .

**2.7. A functoriality property.** Every group homomorphism  $\varphi: G \rightarrow H$  induces a morphism between character groups,  $\widehat{\varphi}: \widehat{H} \rightarrow \widehat{G}$ , given by  $\widehat{\varphi}(\rho)(g) = \varphi(\rho(g))$ . Clearly, the morphism  $\widehat{\varphi}$  sends algebraic subgroups of  $\widehat{H}$  to algebraic subgroups of  $\widehat{G}$ . Moreover, if  $\varphi$  is surjective, then  $\widehat{\varphi}$  is injective. The next lemma indicates a partial functoriality property for the characteristic varieties of groups.

**Lemma 2.13.** *Let  $\varphi: G \twoheadrightarrow Q$  be an epimorphism from a finitely generated group  $G$  to a group  $Q$ . Then the induced monomorphism between character groups,  $\widehat{\varphi}: \widehat{Q} \hookrightarrow \widehat{G}$ , restricts to an embedding  $\mathcal{V}_d^1(Q) \hookrightarrow \mathcal{V}_d^1(G)$ , for each  $d \geq 1$ .*

*Proof.* Let  $\rho: Q \rightarrow \mathbb{C}^\times$  be a character. The 5-term exact sequence associated to the extension  $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$  and the  $\mathbb{Z}G$ -module  $M = \mathbb{C}_{\rho \circ \varphi}$  ends in  $H_1(G, M) \rightarrow H_1(Q, M_K) \rightarrow 0$ , where  $M_K$  denotes the module of coinvariants under the  $K$ -action, see [9, p. 171]. Clearly,  $M_K = \mathbb{C}_\rho$ , as  $\mathbb{Z}Q$ -modules. Hence,  $\dim_{\mathbb{C}} H_1(G, \mathbb{C}_{\rho \circ \varphi})$  is bounded below by  $\dim_{\mathbb{C}} H_1(Q, \mathbb{C}_\rho)$ . Thus, if  $\rho \in \mathcal{V}_d^1(Q)$ , then  $\widehat{\varphi}(\rho) \in \mathcal{V}_d^1(G)$ , and we are done.  $\square$

### 3. A STRATIFICATION OF THE RATIONAL GRASSMANNIAN

In this section, we stratify the Grassmannian of  $r$ -planes in  $H^1(X, \mathbb{Q})$  by certain subsets  $\Omega_r^i(X)$  which keep track of those regular  $\mathbb{Z}^r$ -covers of  $X$  having finite Betti numbers up to degree  $i$ .

**3.1. Free abelian covers.** As before, let  $X$  be a connected CW-complex with finite 1-skeleton, and let  $G = \pi_1(X, x_0)$  be the fundamental group, based at the unique 0-cell  $x_0$ . Following Dwyer and Fried [21], we start by parameterizing in a convenient way the set of all connected, regular covering spaces of  $X$  (up to equivalence of covers), with group of deck transformations a free abelian group of fixed rank  $r$ .

The model situation is the  $r$ -dimensional torus  $T^r = K(\mathbb{Z}^r, 1)$  and its universal cover,  $\mathbb{R}^r \rightarrow T^r$ , with group of deck transformations  $\mathbb{Z}^r$ . Any epimorphism  $\nu: G \twoheadrightarrow \mathbb{Z}^r$  gives rise to a  $\mathbb{Z}^r$ -cover,  $X^\nu \rightarrow X$ , by pulling back the universal cover of  $T^r$  along a map  $f: X \rightarrow T^r$  realizing  $\nu$  at the level of fundamental groups,

$$(22) \quad \begin{array}{ccc} X^\nu & \longrightarrow & \mathbb{R}^r \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & T^r. \end{array}$$

Since this is a pull-back diagram, and  $\mathbb{R}^r$  is contractible, the homotopy fiber of  $f$  is homotopy equivalent to  $X^\nu$ . By covering space theory, all connected, regular  $\mathbb{Z}^r$ -covers of  $X$  arise in the fashion described above.

Now, the homomorphism  $\nu: G \twoheadrightarrow \mathbb{Z}^r$  factors as  $\nu_* \circ \text{ab}$ , where  $\text{ab}: G \rightarrow G_{\text{ab}}$  is the abelianization map, and  $\nu_* = \nu_{\text{ab}}: G_{\text{ab}} \rightarrow \mathbb{Z}^r$  may be identified with the induced homomorphism  $f_*: H_1(X, \mathbb{Z}) \rightarrow H_1(T^r, \mathbb{Z})$ . Passing to the corresponding homomorphism in rational homology, we see that the cover  $X^\nu \rightarrow X$  is determined (up to equivalence) by the kernel of the map  $\nu_*: H_1(X, \mathbb{Q}) \rightarrow \mathbb{Q}^r$ . Conversely, every codimension- $r$  linear subspace of  $H_1(X, \mathbb{Q})$  can be realized as the kernel of  $\nu_*: H_1(X, \mathbb{Q}) \rightarrow \mathbb{Q}^r$ , for some epimorphism  $\nu: G \twoheadrightarrow \mathbb{Z}^r$ , and thus gives rise to a cover  $X^\nu \rightarrow X$ .

Let  $\text{Gr}_r(H^1(X, \mathbb{Q}))$  be the Grassmannian of  $r$ -planes in the finite-dimensional, rational vector space  $H^1(X, \mathbb{Q})$ . Proceeding as above, but using the dual homomorphism  $\nu^*: \mathbb{Q}^r \rightarrow H^1(X, \mathbb{Q})$ , instead, we obtain a one-to-one correspondence between equivalence classes of regular  $\mathbb{Z}^r$ -covers of  $X$  and  $r$ -planes in  $H^1(X, \mathbb{Q})$ , which we record below.

**Proposition 3.1** (Dwyer–Fried [21]). *The connected, regular covers of  $X$  whose group of deck transformations is free abelian of rank  $r$  are parameterized by the rational Grassmannian  $\text{Gr}_r(H^1(X, \mathbb{Q}))$ , via the correspondence*

$$\begin{aligned} \{\text{regular } \mathbb{Z}^r\text{-covers of } X\} &\longleftrightarrow \{r\text{-planes in } H^1(X, \mathbb{Q})\} \\ X^\nu \rightarrow X &\longleftrightarrow P_\nu := \text{im}(\nu^*) \end{aligned}$$

This correspondence enjoys a nice functoriality property. Let  $f: (X, x_0) \rightarrow (Y, y_0)$  be a pointed map, and denote by  $f_\sharp: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$  the induced homomorphism on fundamental groups.

**Lemma 3.2.** *Suppose  $f_\sharp: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ , is surjective. If  $\nu: \pi_1(Y, y_0) \twoheadrightarrow \mathbb{Z}^r$  is an epimorphism, then  $P_{\nu \circ f_\sharp} = f^*(P_\nu)$ .*

*Proof.* By the lifting criterion, the map  $f$  lifts to a map  $\bar{f}$  between the  $\mathbb{Z}^r$ -covers defined by the epimorphisms  $\nu \circ f_\sharp$  and  $\nu$ . We then have a pullback diagram,

$$(23) \quad \begin{array}{ccc} X^{\nu \circ f_\sharp} & \xrightarrow{\bar{f}} & Y^\nu \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

Clearly, the morphism  $f_*: H_1(X, \mathbb{Q}) \rightarrow H_1(Y, \mathbb{Q})$  is surjective. Thus, the morphism  $f^*: H^1(Y, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q})$  is injective, and takes the  $r$ -plane  $P_\nu = \text{im}(\nu^*)$  to the  $r$ -plane  $f^*(P_\nu) = \text{im}(f^* \circ \nu^*)$ , which coincides with  $\text{im}((\nu \circ f_\sharp)^*) = P_{\nu \circ f_\sharp}$ .  $\square$

**3.2. The Dwyer–Fried invariants.** Moving about the rational Grassmannian, and keeping track of how the Betti numbers of the corresponding covers vary leads to the following definition (see [21], and also [35], [44]).

**Definition 3.3.** The *Dwyer–Fried invariants* of  $X$  are the subsets

$$\Omega_r^i(X) = \{P_\nu \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid b_j(X^\nu) < \infty \text{ for } j \leq i\},$$

defined for all  $i \geq 0$  and all  $r \geq 1$ , with the convention that  $\Omega_r^i(X) = \text{Gr}_r(H^1(X, \mathbb{Q})) = \emptyset$  if  $r > b_1(X)$ .

If  $b_1(X) = 0$ , then all the  $\Omega$ -invariants of  $X$  are empty. For a fixed  $r \geq 1$ , the Dwyer–Fried invariants form a descending filtration of the Grassmannian of  $r$ -planes,

$$(24) \quad \text{Gr}_r(H^1(X, \mathbb{Q})) = \Omega_r^0(X) \supseteq \Omega_r^1(X) \supseteq \Omega_r^2(X) \supseteq \cdots,$$

with intersection  $\Omega_r(X) = \{P_\nu \mid \dim_{\mathbb{Q}} H_*(X^\nu, \mathbb{Q}) < \infty\}$ .

**Remark 3.4.** Dwyer and Fried only consider finite CW-complexes  $X$ , and the sets  $\Omega_r(X)$ . Of course, if  $\dim X = k$ , then  $\Omega_r(X) = \Omega_r^k(X)$ . We prefer to work with the filtration (24), which provides more refined information on the  $\mathbb{Z}^r$ -covers of  $X$ .

The  $\Omega$ -sets are homotopy-type invariants of  $X$ . More precisely, we have the following result.

**Lemma 3.5.** *Suppose  $X$  is homotopy equivalent to  $Y$ . For each  $r \geq 1$ , there is an isomorphism  $\text{Gr}_r(H^1(Y, \mathbb{Q})) \cong \text{Gr}_r(H^1(X, \mathbb{Q}))$  which sends each subset  $\Omega_r^i(Y)$  bijectively onto  $\Omega_r^i(X)$ .*

*Proof.* Let  $f: X \rightarrow Y$  be a (cellular) homotopy equivalence. The induced isomorphism in rational cohomology defines isomorphisms  $f_r^*: \text{Gr}_r(H^1(Y, \mathbb{Q})) \rightarrow \text{Gr}_r(H^1(X, \mathbb{Q}))$  between the corresponding Grassmannians. It remains to show that  $f_r^*(\Omega_r^i(Y)) \subseteq \Omega_r^i(X)$ . For that, let  $P \in \Omega_r^i(Y)$ , and write  $P = P_\nu$ , for some epimorphism  $\nu: \pi_1(Y) \twoheadrightarrow \mathbb{Z}^r$ . The map  $f$  lifts to a map  $\bar{f}: X^{\nu \circ f_\#} \rightarrow Y^\nu$  as in (23). Clearly,  $\bar{f}$  is a homotopy equivalence. Thus,  $b_i(X^{\nu \circ f_\#}) = b_i(Y^\nu)$ , and so  $f_r^*(P_\nu) = P_{\nu \circ f_\#}$  belongs to  $\Omega_r^i(X)$ .  $\square$

In view of this lemma, we may extend the definition of the  $\Omega$ -sets from spaces to groups. Let  $G$  be a finitely-generated group. Pick a classifying space  $K(G, 1)$  with finite  $k$ -skeleton, for some  $k \geq 1$ .

**Definition 3.6.** The *Dwyer–Fried invariants* of a group  $G$  are the subsets  $\Omega_r^i(G) = \Omega_r^i(K(G, 1))$  of  $\text{Gr}_r(H^1(G, \mathbb{Q}))$ , defined for all  $i \geq 0$  and  $r \geq 1$ .

Since the homotopy type of  $K(G, 1)$  depends only on  $G$ , the sets  $\Omega_r^i(G)$  are well-defined group invariants.

**3.3. Discussion.** Especially manageable is the situation when  $n = b_1(X) > 0$  and  $r = n$ . In this case,  $\text{Gr}_n(H^1(X, \mathbb{Q})) = \{\text{pt}\}$ . Under the correspondence from Proposition 3.1, this single point is realized by the maximal free abelian cover,  $X^\alpha \rightarrow X$ , where  $\alpha: G \twoheadrightarrow G_{\text{ab}}/\text{Tors}(G_{\text{ab}}) = \mathbb{Z}^n$  is the canonical projection. We then have

$$(25) \quad \Omega_n^i(X) = \begin{cases} \{\text{pt}\} & \text{if } b_j(X^\alpha) < \infty \text{ for all } j \leq i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Both situations may occur, as illustrated by a very simple example.

**Example 3.7.** Let  $X = S^1 \vee S^k$ , for some  $k > 1$ . Then  $X^\alpha$  is homotopic to a countable wedge of  $k$ -spheres. Thus,  $\Omega_1^i(X) = \{\text{pt}\}$  for  $i < k$ , yet  $\Omega_1^i(X) = \emptyset$ , for  $i \geq k$ .

It should be emphasized that finiteness of the Betti numbers of a free abelian cover  $X^\nu$  does not necessarily imply finite-generation of the integral homology groups of  $X^\nu$ . Thus, we cannot replace in Definition 3.3 the condition “ $b_i(X^\nu) < \infty$ , for  $i \leq q$ ” by the (stronger) condition “ $H_i(X^\nu, \mathbb{Z})$  is a finitely-generated group, for  $i \leq q$ .” Example 3.8 below (extracted from a paper of Milnor [33]) explains why.

**Example 3.8.** Let  $K$  be a knot in  $S^3$ , with complement  $X = S^3 \setminus K$ , and infinite cyclic cover  $X^{\text{ab}}$ . As is well-known,  $H_1(X^{\text{ab}}, \mathbb{Z}) = \mathbb{Z}[t^{\pm 1}]/(\Delta_K)$ , where  $\Delta_K$  is the Alexander polynomial of  $K$ . Hence,  $H_1(X^{\text{ab}}, \mathbb{Q}) = \mathbb{Q}^d$ , where  $d$  is the degree of  $\Delta_K$ , and so  $\Omega_1^1(X) = \{\text{pt}\}$ . On the other hand, if the Alexander polynomial is not monic,  $H_1(X^{\text{ab}}, \mathbb{Z})$  need not be finitely generated as a  $\mathbb{Z}$ -module. For instance, let  $K$  be the  $5_2$  knot, with Alexander polynomial  $\Delta_K = 2t^2 - 3t + 2$ . Then  $H_1(X^{\text{ab}}, \mathbb{Z}) = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$  is not finitely generated, though, of course,  $H_1(X^{\text{ab}}, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$ .

## 4. DWYER–FRIED INVARIANTS FROM CHARACTERISTIC VARIETIES

In this section, we explain how the characteristic varieties of a space  $X$  determine the  $\Omega$ -invariants of  $X$ , and therefore control the homological finiteness properties of its (regular) free abelian covers.

**4.1. Homological finiteness and characteristic varieties.** Given an epimorphism  $\nu: G \twoheadrightarrow \mathbb{Z}^r$ , let  $\hat{\nu}: \widehat{\mathbb{Z}^r} \hookrightarrow \widehat{G}$  be the induced homomorphism between character groups. Its image,  $T_\nu = \hat{\nu}(\widehat{\mathbb{Z}^r})$ , is a complex algebraic subtorus of  $\widehat{G}$ , isomorphic to  $(\mathbb{C}^\times)^r$ .

The following theorem was proved by Dwyer and Fried in [21] for finite CW-complexes, using the support loci for the Alexander invariants of such spaces. It was recast in a slightly more general context by Papadima and Suciu in [35], using the degree-1 characteristic varieties.

**Theorem 4.1** ([21], [35]). *Let  $X$  be a connected CW-complex with finite  $k$ -skeleton, for some  $k \geq 1$ , and let  $G = \pi_1(X)$ . For an epimorphism  $\nu: G \twoheadrightarrow \mathbb{Z}^r$ , the following are equivalent:*

- (1) *The vector space  $\bigoplus_{i=0}^k H_i(X^\nu, \mathbb{C})$  is finite-dimensional.*
- (2) *The algebraic torus  $T_\nu$  intersects the variety  $\mathcal{V}^k(X)$  in only finitely many points.*

In other words, whether all the Betti numbers  $b_1(X^\nu), \dots, b_k(X^\nu)$  are finite or not is dictated by whether the variety  $T_\nu \cap \mathcal{V}^k(X)$  has dimension 0 or not. More generally, it is shown in [48] that the varieties  $\mathcal{V}^k(X)$  control the finiteness of the Betti numbers of all abelian (not necessarily torsion-free) regular covers of the space  $X$ .

**4.2. Reinterpreting the  $\Omega$ -invariants.** Let  $\exp: H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathbb{C}^\times)$  be the coefficient homomorphism induced by the homomorphism  $\mathbb{C} \rightarrow \mathbb{C}^\times, z \mapsto e^z$ .

**Lemma 4.2.** *Let  $\nu: G \twoheadrightarrow \mathbb{Z}^r$  be an epimorphism. Under the universal coefficient isomorphism  $H^1(X, \mathbb{C}^\times) \cong \text{Hom}(G, \mathbb{C}^\times)$ , the complex  $r$ -torus  $\exp(P_\nu \otimes \mathbb{C})$  corresponds to  $T_\nu = \hat{\nu}(\widehat{\mathbb{Z}^r})$ .*

*Proof.* By naturality of the coefficient homomorphism and the universal coefficients isomorphism, the following diagram commutes.

$$\begin{array}{ccccc}
 \mathbb{Q}^r & \xrightarrow{\nu^*} & H^1(X, \mathbb{Q}) & & \\
 \downarrow & & \downarrow & & \\
 \text{Hom}(\mathbb{Z}^r, \mathbb{C}) \cong \mathbb{C}^r & \xrightarrow{\nu^*} & H^1(X, \mathbb{C}) \cong \text{Hom}(G, \mathbb{C}) & & \\
 \downarrow \text{Hom}(\cdot, \exp) & \searrow \exp & \downarrow \exp & & \downarrow \text{Hom}(\cdot, \exp) \\
 \text{Hom}(\mathbb{Z}^r, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^r & \xrightarrow{\nu^*} & H^1(X, \mathbb{C}^\times) \cong \text{Hom}(G, \mathbb{C}^\times) & & \\
 & \searrow \hat{\nu} = \text{Hom}(\nu, \cdot) & & & 
 \end{array}$$

By definition,  $P_\nu$  is the image of the top  $\nu^*$  map. Hence,  $P_\nu \otimes \mathbb{C}$  is the image of the middle  $\nu^*$  map, while  $\exp(P_\nu \otimes \mathbb{C})$  is the image of the shaded arrow. By surjectivity of  $\exp: \mathbb{C}^r \rightarrow (\mathbb{C}^\times)^r$ , this last image is the same as the image of the bottom  $\nu^*$  map, which in turn corresponds to  $T_\nu = \text{im}(\hat{\nu})$  under the bottom-right isomorphism.  $\square$

Recall we introduced in §2.5 the varieties  $\mathcal{W}^k(X) = \mathcal{V}^k(X) \cap \widehat{G}^\circ$ , lying in the identity component of the character group  $\widehat{G} = H^1(G, \mathbb{C}^\times)$ .

**Theorem 4.3.** *Let  $X$  be a CW-complex with finite  $k$ -skeleton. Then, for all  $i \leq k$  and  $r \geq 1$ ,*

$$\Omega_r^i(X) = \{P \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid \#(\exp(P \otimes \mathbb{C}) \cap \mathcal{W}^i(X)) < \infty\}.$$

*Proof.* Let  $P$  be a rational  $r$ -plane in  $H^1(X, \mathbb{Q})$ . By Proposition 3.1, there is an epimorphism  $\nu: G \twoheadrightarrow \mathbb{Z}^r$  such that  $P = P_\nu$ . Clearly, the algebraic torus  $T_\nu$  lies in  $\widehat{G}^\circ$ ; thus,  $T_\nu \cap \mathcal{V}^i(X) = T_\nu \cap \mathcal{W}^i(X)$ . Applying Theorem 4.1 and Lemma 4.2 finishes the proof.  $\square$

In other words, an  $r$ -plane  $P_\nu \in \text{Gr}_r(H^1(X, \mathbb{Q}))$  belongs to  $\Omega_r^i(X)$  if and only if there are only finitely many characters  $\rho: \mathbb{Z}^r \rightarrow \mathbb{C}^\times$  such that  $b_j(X, \rho \circ \nu) \neq 0$ , for some  $j \leq i$ .

**4.3. Some applications.** In the case when  $r = b_1(X)$ , Theorem 4.3 allows us to improve on the discussion from §3.3 regarding the Betti numbers of maximal free abelian covers.

**Theorem 4.4.** *Let  $X$  be a CW-complex with finite  $k$ -skeleton, and set  $n = b_1(X)$ . Let  $X^\alpha \rightarrow X$  be the maximal free abelian cover. For each  $i \leq k$ , the following are equivalent:*

- (1)  $b_j(X^\alpha) < \infty$ , for all  $j \leq i$ .
- (2)  $\Omega_n^i(X) \neq \emptyset$ .
- (3)  $\mathcal{W}^i(X)$  is finite.

*Proof.* We may assume  $n > 0$ , since, otherwise, there is nothing to prove. The only  $n$ -plane in  $H = H^1(X, \mathbb{Q})$  is  $H$  itself; thus,  $\text{Gr}_n(H)$  consists of a single point, which corresponds to the cover  $X^\alpha \rightarrow X$  given by the projection  $\alpha: \pi_1(X) \twoheadrightarrow \mathbb{Z}^n$ .

By formula (25), conditions (1) and (2) are equivalent. On the other hand,  $\exp(H \otimes \mathbb{C}) \cap \mathcal{W}^i(X) = \mathcal{W}^i(X)$ ; thus, Theorem 4.3 shows that conditions (2) and (3) are equivalent.  $\square$

The following particular case is worth singling out.

**Corollary 4.5.** *Assume  $X$  has finite 1-skeleton. Then  $\mathcal{W}^1(X)$  is finite if and only if  $b_1(X^\alpha)$  is finite.*

**4.4. Large  $\Omega$ -invariants.** We now analyze the situation when the Dwyer–Fried sets comprise the whole Grassmannian. The next result follows at once from Theorem 4.3.

**Proposition 4.6.** *If  $\mathcal{W}^i(X)$  is a finite set, then  $\Omega_r^i(X) = \text{Gr}_r(H^1(X, \mathbb{Q}))$ , for all  $r \geq 1$ .*

A general class of examples is provided by nilmanifolds.

**Example 4.7.** Let  $G$  be a torsion-free, finitely generated nilpotent group. Then  $G$  admits as classifying space a compact nilmanifold of the form  $M = \mathbb{R}^n/G$ . Furthermore,  $\mathcal{V}_d^i(M)$  equals  $\{1\}$  if  $d \leq b_i(M)$  and is empty, otherwise. (This fact was established in [32], using the Hochschild–Serre spectral sequence and induction on the nilpotency class of  $G$ .) It follows that  $\Omega_r^i(M) = \text{Gr}_r(H^1(M, \mathbb{Q}))$ , for all  $i \geq 0$  and  $r \geq 1$ .

In particular,  $\Omega_r^i(T^n) = \text{Gr}_r(\mathbb{Q}^n)$ , reflecting the fact that every connected cover of the  $n$ -torus is homotopy equivalent to a  $k$ -torus, for some  $0 \leq k \leq n$ .

Further examples are provided by knot complements.

**Proposition 4.8.** *Let  $K_1, \dots, K_n$  be codimension-2 spheres smoothly embedded in  $S^m$ , for some  $m \geq 3$ , and let  $X = X_1 \times \dots \times X_n$  be the product of the respective complements. Then  $\Omega_r^i(X) = \text{Gr}_r(\mathbb{Q}^n)$ , for all  $i \geq 0$ .*

*Proof.* First assume  $n = 1$ , and let  $X$  be a knot complement. The homology groups of the infinite cyclic cover  $X^{\text{ab}}$  with coefficients in  $\mathbb{C}$  are modules over the principal ideal domain  $\Lambda = \mathbb{C}[t^{\pm 1}]$ . A spectral sequence argument (due to J. Levine) shows that  $t - 1$

acts invertibly on these Alexander modules. It follows that  $\bigoplus_{j \leq i} H_j(X^{\text{ab}}, \mathbb{C})$  is a torsion  $\Lambda$ -module, for each  $i \geq 0$ ; hence, its support  $\mathcal{W}^i(X)$  is a finite subset of  $\mathbb{C}^\times$ .

For the general case, we know from the above argument that the sets  $\mathcal{W}^i(X_1), \dots, \mathcal{W}^i(X_n)$  are finite. From Corollary 2.10, we deduce that  $\mathcal{W}^i(X)$  is finite, for each  $i \geq 0$ . The desired conclusion follows from Proposition 4.6.  $\square$

**4.5. Empty  $\Omega$ -invariants.** At the other extreme, we have a rather large supply of spaces for which the Dwyer–Fried sets are empty.

**Proposition 4.9.** *Suppose  $\mathcal{V}_1^j(X) = H^1(X, \mathbb{C}^\times)$ , for some  $j > 0$ . Then  $\Omega_r^i(X) = \emptyset$ , for all  $i \geq j$  and all  $r \geq 1$ .*

*Proof.* If  $r > b_1(X)$ , then of course  $\Omega_r^i(X) = \emptyset$ . Otherwise, we may use Theorem 4.3. By hypothesis,  $\mathcal{V}^i(X) = H^1(X, \mathbb{C}^\times)$ , for all  $i \geq j$ . Hence, for every rational  $r$ -plane  $P \subset H^1(X, \mathbb{Q})$ , the intersection  $\exp(P \otimes \mathbb{C}) \cap \mathcal{V}^i(X)$  is isomorphic to  $(\mathbb{C}^\times)^r$ , which is an infinite set. Thus,  $\Omega_r^i(X) = \emptyset$ .  $\square$

From Propositions 2.11 and 4.9, we obtain the following corollary.

**Corollary 4.10.** *Let  $X = X_1 \vee \dots \vee X_n$ , and suppose  $b_1(X_s) > 0$ , for all  $s$ . Then  $\Omega_r^i(X) = \emptyset$ , for all  $i, r \geq 1$ .*

Similarly, using Propositions 2.9 and 4.9, we obtain the following corollary.

**Corollary 4.11.** *Let  $X = X_1 \times \dots \times X_n$ , and suppose that  $\mathcal{V}_1^1(X_s) = H^1(X_s, \mathbb{C}^\times)$ , for all  $s$ . Then  $\Omega_r^i(X) = \emptyset$ , for all  $i \geq n$  and  $r \geq 1$ .*

**Example 4.12.** Let  $\Sigma_g$  be a Riemann surface of genus  $g > 1$ . From Example 2.7, we know that  $\mathcal{V}_1^1(\Sigma_g) = H^1(\Sigma_g, \mathbb{C}^\times)$ . Proposition 4.9 now gives  $\Omega_r^i(\Sigma_g) = \emptyset$ , for all  $i, r \geq 1$ . For a product of surfaces, Corollary 4.11 yields

$$(26) \quad \Omega_r^n(\Sigma_{g_1} \times \dots \times \Sigma_{g_n}) = \emptyset, \quad \text{for all } r \geq 1.$$

**Example 4.13.** Let  $Y_m = \bigvee^m S^1$  be a wedge of  $m$  circles,  $m > 1$ . From Example 2.6, we know that  $\mathcal{V}_1^1(Y_m) = H^1(Y_m, \mathbb{C}^\times)$ . By Proposition 4.9 (or Corollary 4.10), we have  $\Omega_r^i(Y_m) = \emptyset$ , for all  $i, r \geq 1$ . For a product of wedges of circles, Corollary 4.11 yields

$$(27) \quad \Omega_r^n(Y_{m_1} \times \dots \times Y_{m_n}) = \emptyset, \quad \text{for all } r \geq 1.$$

**4.6. A naturality property.** We saw in Lemma 2.13 that the characteristic varieties enjoy a nice naturality property with respect to epimorphisms between finitely generated groups. As we shall see in Example 4.15, the analogous property does not hold for the Dwyer–Fried sets. Nevertheless, it does hold for their complements.

**Proposition 4.14.** *Let  $\varphi: G \twoheadrightarrow Q$  be an epimorphism from a finitely generated group  $G$  to a group  $Q$ , and let  $\varphi^*: H^1(Q, \mathbb{Q}) \hookrightarrow H^1(G, \mathbb{Q})$  be the induced monomorphism in cohomology. Then, for each  $r \geq 1$ , the corresponding morphism between Grassmannians,  $\varphi_r^*: \text{Gr}_r(H^1(Q, \mathbb{Q})) \hookrightarrow \text{Gr}_r(H^1(G, \mathbb{Q}))$ , restricts to an embedding  $\Omega_r^1(Q)^\complement \hookrightarrow \Omega_r^1(G)^\complement$ .*

*Proof.* Let  $P = P_\nu$  be a plane in  $\text{Gr}_r(H^1(Q, \mathbb{Q})) \setminus \Omega_r^1(Q)$ . By Theorem 4.3, the intersection of  $\exp(P \otimes \mathbb{C})$  with  $\mathcal{W}^1(Q)$  is infinite. By Lemma 2.13, the morphism  $\hat{\varphi}: \hat{Q} \hookrightarrow \hat{G}$  restricts to an embedding  $\mathcal{W}^1(Q) \hookrightarrow \mathcal{W}^1(G)$ . By Lemma 3.2, we have that  $\varphi_r^*(P_\nu) = P_{\hat{\varphi}(\nu)}$ . Hence, the intersection of  $\exp(\varphi_r^*(P) \otimes \mathbb{C})$  with  $\mathcal{W}^1(G)$  is infinite. Thus,  $\varphi_r^*(P)$  belongs to  $\text{Gr}_r(H^1(G, \mathbb{Q})) \setminus \Omega_r^1(G)$ .  $\square$



**Example 4.15.** Let  $F_n$  be the free group of rank  $n > 1$ . The abelianization map,  $\text{ab}: F_n \rightarrow \mathbb{Z}^n$ , induces isomorphisms  $\text{ab}_r^*: \text{Gr}_r(H^1(\mathbb{Z}^n, \mathbb{Q})) \xrightarrow{\cong} \text{Gr}_r(H^1(F_n, \mathbb{Q}))$ , for all  $r \geq 1$ . Corollary 4.7 shows that  $\Omega_r^1(\mathbb{Z}^n) = \text{Gr}_r(\mathbb{Q}^n)$ , whereas Example 4.13 shows that  $\Omega_r^1(F_n) = \emptyset$ . Thus, the map  $\text{ab}_r^*$  does *not* restrict to an embedding  $\Omega_r^1(\mathbb{Z}^n) \hookrightarrow \Omega_r^1(F_n)$ , though of course it restricts to an embedding  $\Omega_r^1(\mathbb{Z}^n)^{\text{e}} \hookrightarrow \Omega_r^1(F_n)^{\text{e}}$ .

## 5. CHARACTERISTIC SUBSPACE ARRANGEMENTS

In this section, we associate to each space  $X$  a sequence of subspace arrangements,  $\mathcal{C}_i(X)$ , all lying in the rational vector space  $H^1(X, \mathbb{Q})$ , and providing a rough linear approximation to the characteristic varieties  $\mathcal{W}^i(X)$ .

**5.1. The exponential tangent cone to a variety.** We start by reviewing a notion introduced by Dimca, Papadima and Suciu in [20], a notion that will play an essential role for the rest of this paper.

**Definition 5.1.** Let  $W$  be an algebraic subset of the complex algebraic torus  $(\mathbb{C}^\times)^n$ . The *exponential tangent cone* of  $W$  at the identity 1 is the homogeneous subvariety  $\tau_1(W)$  of  $\mathbb{C}^n$ , given by

$$(28) \quad \tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \text{ for all } \lambda \in \mathbb{C}\}.$$

The terminology reflects certain similarities between the exponential tangent cone  $\tau_1(W)$  and the classical tangent cone  $\text{TC}_1(W)$ . As noted in [20], the inclusion  $\tau_1(W) \subseteq \text{TC}_1(W)$  always holds, but not necessarily as an equality. For more on this, we refer to [35] and [45]. We illustrate the construction with a simple example.

**Example 5.2.** Let  $W = \{t \in (\mathbb{C}^\times)^2 \mid t_1 + t_2 = 2\}$ . The set  $\tau_1(W)$  consists of all pairs  $(z_1, z_2) \in \mathbb{C}^2$  for which  $e^{\lambda z_1} + e^{\lambda z_2} = 2$ , for all  $\lambda \in \mathbb{C}$ . Expanding in Taylor series around 0, we find that  $z_1^2 + z_2^2 = 0$ , and thus  $\tau_1(W) = \{0\}$ .

It is readily seen that  $\tau_1$  commutes with finite unions and arbitrary intersections. Clearly, the exponential tangent cone of  $W$  only depends on the analytic germ of  $W$  at the identity  $1 \in (\mathbb{C}^\times)^n$ . In particular,  $\tau_1(W) \neq \emptyset$  if and only if  $1 \in W$ . The following diagram summarizes the situation:

$$\begin{array}{ccc} 0 \in \tau_1(W) & \hookrightarrow & \mathbb{C}^n \\ \downarrow & \downarrow \text{exp} & \downarrow \text{exp} \\ 1 \in W & \hookrightarrow & (\mathbb{C}^\times)^n. \end{array}$$

**Example 5.3.** Suppose  $T$  is an algebraic  $r$ -subtorus of  $(\mathbb{C}^\times)^n$ . Then  $T = \exp(P \otimes \mathbb{C})$ , for some  $r$ -dimensional subspace  $P$  inside  $\mathbb{Q}^n$ . It follows that  $\tau_1(T) = P \otimes \mathbb{C}$ , a rationally defined linear subspace of  $\mathbb{C}^n$ . Hence,  $\tau_1(T)$  coincides with  $T_1(T)$ , the tangent space at the identity to the Lie group  $T$ .

In fact, as shown in [20], a much more general result holds: *every* exponential tangent cone  $\tau_1(W)$  is a union of linear subspaces defined over  $\mathbb{Q}$ . Since the proof given in [20] is rather sketchy, we will provide full details in Lemma 5.4 and Theorem 5.5 below. Along the way, we shall outline an explicit computational algorithm for determining  $\tau_1(W)$ , for any algebraic subvariety  $W \subset (\mathbb{C}^\times)^n$ .

**5.2. The structure of exponential tangent cones.** To start with, fix a non-zero Laurent polynomial  $f \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$  with  $f(1) = 0$ . Write

$$(29) \quad f(t_1, \dots, t_n) = \sum_{a \in S} c_a t_1^{a_1} \cdots t_n^{a_n},$$

where  $S$  is a finite subset of  $\mathbb{Z}^n$ , and  $c_a \neq 0$  for each  $a = (a_1, \dots, a_n) \in S$ . We say a partition  $\mathbf{p} = (\mathbf{p}_1 \mid \cdots \mid \mathbf{p}_q)$  of the support  $S$  is *admissible* if  $\sum_{a \in \mathbf{p}_i} c_a = 0$ , for all  $1 \leq i \leq q$ . In particular, the trivial partition  $\mathbf{p} = (S)$  is admissible, by our assumption on  $f$ .

To a partition  $\mathbf{p}$  as above we associate a linear subspace  $L(\mathbf{p}) \subset \mathbb{Q}^n$ , defined as

$$(30) \quad L(\mathbf{p}) = \{x \in \mathbb{Q}^n \mid (a - b) \cdot x = 0, \forall a, b \in \mathbf{p}_i, \forall 1 \leq i \leq q\}.$$

Given a vector  $z \in \mathbb{C}^n$ , define an analytic function  $\phi_z: \mathbb{C} \rightarrow \mathbb{C}$  by

$$(31) \quad \phi_z(\lambda) = f(e^{\lambda z_1}, \dots, e^{\lambda z_n}) = \sum_{a \in S} c_a e^{(a \cdot z)\lambda},$$

where  $a \cdot z = \sum_{i=1}^n a_i z_i$  is the standard dot product.

**Lemma 5.4.** *The function  $\phi_z$  vanishes identically if and only if  $z$  belongs to a linear subspace of the form  $L(\mathbf{p}) \otimes \mathbb{C}$ , where  $\mathbf{p}$  is a admissible partition of  $S$ .*

*Proof.* Suppose  $z \in L(\mathbf{p}) \otimes \mathbb{C}$ . Then, for each part  $\mathbf{p}_i$ , there is a constant  $k_i$  such that  $a \cdot z = k_i$  for all  $a \in \mathbf{p}_i$ . Write  $f_i = \sum_{a \in \mathbf{p}_i} c_a t_1^{a_1} \cdots t_n^{a_n}$ , and let  $\phi_z^i$  be the corresponding analytic function. We then have

$$\phi_z^i(\lambda) = \sum_{a \in \mathbf{p}_i} c_a e^{(a \cdot z)\lambda} = \sum_{a \in \mathbf{p}_i} c_a e^{k_i \lambda} = \left( \sum_{a \in \mathbf{p}_i} c_a \right) e^{k_i \lambda},$$

which equals 0, since  $\mathbf{p}$  is admissible. On the other hand,  $\phi_z = \sum_{i=1}^q \phi_z^i$ . Thus,  $\phi_z \equiv 0$ .

Conversely, suppose  $\phi_z \equiv 0$ . Since  $f$  is non-zero, we can find a maximal subset  $\mathbf{p}_1 \subset S$  such that  $a \cdot z$  is constant for all  $a \in \mathbf{p}_1$ . Replacing  $S$  by  $S \setminus \mathbf{p}_1$  and proceeding in this fashion, we ultimately arrive at a partition  $\mathbf{p} = (\mathbf{p}_1 \mid \cdots \mid \mathbf{p}_q)$ , and pairwise distinct constants  $k_1, \dots, k_q$  such that  $a \cdot z = k_i$ , for all  $a \in \mathbf{p}_i$ . Thus,  $z \in L(\mathbf{p}) \otimes \mathbb{C}$ . On the other hand, the identity  $\phi_z = \sum_{i=1}^q \phi_z^i \equiv 0$ , translates into a linear dependence,

$$\sum_{i=1}^q \left( \sum_{a \in \mathbf{p}_i} c_a \right) e^{k_i \lambda} = 0, \quad \forall \lambda \in \mathbb{C}.$$

Computing the Wronskian of  $e^{k_1 \lambda}, \dots, e^{k_q \lambda}$  reveals that these functions are linearly independent. Hence,  $\sum_{a \in \mathbf{p}_i} c_a = 0$ , for all  $i$ , showing that  $\mathbf{p}$  is an admissible partition.  $\square$

**Theorem 5.5** ([20]). *The exponential tangent cone  $\tau_1(W)$  is a finite union of rationally defined linear subspaces of  $\mathbb{C}^n$ .*

*Proof.* The variety  $W \subset (\mathbb{C}^\times)^n$  is the common zero-locus of finitely many Laurent polynomials in  $n$  variables, say,  $f_1, \dots, f_m$ . Since  $\tau_1$  commutes with intersections, we have that  $\tau_1(W) = \bigcap_{j=1}^m \tau_1(V(f_j))$ . Thus, it is enough to consider the case  $W = V(f)$ , where  $f$  is a non-zero Laurent polynomial. Without loss of generality, we may assume  $f(1) = 0$ , for otherwise  $\tau_1(W) = \emptyset$ .

By definition, a point  $z \in \mathbb{C}^n$  belongs to  $\tau_1(W)$  if and only if the function  $\phi_z$  defined in (31) vanishes identically. By Lemma 5.4, this happens precisely when  $z$  belong to a linear subspace of the form  $L(\mathbf{p}) \otimes \mathbb{C}$ , with  $\mathbf{p}$  an admissible partition. Hence,  $\tau_1(W)$  is the union of all such (rationally defined) subspaces.  $\square$

Note that only maximal partitions contribute to the above union. Indeed, if  $\mathbf{p}$  is a refinement of  $\mathbf{p}'$ , then  $L(\mathbf{p}) \supseteq L(\mathbf{p}')$ .

**5.3. A class of rational subspace arrangements.** As usual, let  $X$  be a connected CW-complex with finite  $k$ -skeleton. Set  $n = b_1(X)$ , and identify  $H^1(X, \mathbb{C}) = \mathbb{C}^n$  and  $H^1(X, \mathbb{C}^\times)^\circ = (\mathbb{C}^\times)^n$ . Let us apply the exponential tangent cone construction to the characteristic varieties  $\mathcal{W}^i(X) \subseteq (\mathbb{C}^\times)^n$ . In view of Theorem 5.5, we may make the following definition.

**Definition 5.6.** For each  $i \leq k$ , the  $i$ -th characteristic arrangement of  $X$ , denoted  $\mathcal{C}_i(X)$ , is the subspace arrangement in  $H^1(X, \mathbb{Q})$  whose complexified union is the exponential tangent cone to  $\mathcal{W}^i(X)$ :

$$(32) \quad \tau_1(\mathcal{W}^i(X)) = \bigcup_{L \in \mathcal{C}_i(X)} L \otimes \mathbb{C}.$$

Put differently, the set of rational points on the exponential tangent cone to  $\mathcal{W}^i(X)$ ,

$$(33) \quad \tau_1^\mathbb{Q}(\mathcal{W}^i(X)) = \tau_1(\mathcal{W}^i(X)) \cap H^1(X, \mathbb{Q}),$$

equals the union of the subspaces comprising the  $i$ -th characteristic arrangement. We thus have a sequence  $\mathcal{C}_0(X), \mathcal{C}_1(X), \dots, \mathcal{C}_k(X)$  of rational subspace arrangements, all lying in the same affine space  $H^1(X, \mathbb{Q}) = \mathbb{Q}^n$ .

From Lemma 2.4, we easily see that the subspace arrangements  $\mathcal{C}_i(X)$  depend only on the homotopy type of  $X$ . Using now Lemma 2.13, we obtain the following consequence.

**Proposition 5.7.** *Let  $\varphi: G \rightarrow Q$  be an epimorphism from a finitely generated group  $G$  to a group  $Q$ . The induced morphism in cohomology,  $\varphi^*: H^1(Q, \mathbb{Q}) \hookrightarrow H^1(G, \mathbb{Q})$ , restricts to a map  $\varphi^*: \tau_1^\mathbb{Q}(Q) \hookrightarrow \tau_1^\mathbb{Q}(G)$ , whose image coincides with  $\tau_1^\mathbb{Q}(\varphi(\mathcal{W}^1(Q)))$ .*

**Example 5.8.** Let  $L$  be the closed three-link chain (the link  $6_1^3$  from Rolfsen's tables [37]), with Alexander polynomial  $\Delta_L = t_1 + t_2 + t_3 - t_1 t_2 - t_1 t_3 - t_2 t_3$ . Let  $X = S^3 \setminus L$ , and fix a meridional basis  $e_1, e_2, e_3$  for  $H_1(X, \mathbb{Z}) = \mathbb{Z}^3$ . From Example 2.8, we know that  $\mathcal{W}^1(X)$  is the zero-locus of  $\Delta_L$ . A straightforward computation show that the support of this polynomial,  $S = \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3\}$ , has precisely three (maximal) admissible partitions, namely,  $\mathbf{p} = (e_1, e_2 + e_3 \mid e_2, e_1 + e_2 \mid e_3, e_1 + e_3)$ ,  $\mathbf{p}' = (e_1, e_1 + e_2 \mid e_2, e_1 + e_3 \mid e_3, e_2 + e_3)$ , and  $\mathbf{p}'' = (e_1, e_1 + e_3 \mid e_2, e_2 + e_3 \mid e_3, e_1 + e_2)$ . Therefore, the arrangement  $\mathcal{C}_1(X)$  consists of three lines in  $\mathbb{Q}^3$ , to wit,  $L(\mathbf{p}) = \{x_1 = x_2 + x_3 = 0\}$ ,  $L(\mathbf{p}') = \{x_2 = x_1 + x_3 = 0\}$ , and  $L(\mathbf{p}'') = \{x_3 = x_1 + x_2 = 0\}$ .

## 6. AN UPPER BOUND FOR THE $\Omega$ -INVARIANTS

In this section, we give a readily computable approximation to the Dwyer–Fried sets, based on the exponential tangent cone construction described above.

**6.1.  $\Omega$ -invariants and characteristic arrangements.** As usual, let  $X$  be a connected CW-complex with finite  $k$ -skeleton, for some  $k \geq 1$ . For each  $i \leq k$ , denote by  $\mathcal{C}_i(X)$  the rational subspace arrangement defined by (32).

**Theorem 6.1.** *For all  $i \leq k$  and all  $r \geq 1$ , we have:*

$$(34) \quad \Omega_r^i(X) \subseteq \left( \bigcup_{L \in \mathcal{C}_i(X)} \{P \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid P \cap L \neq \{0\}\} \right)^\circ.$$

*Proof.* Fix an  $r$ -plane  $P$  inside  $H^1(X, \mathbb{Q})$ , and let  $T = \exp(P \otimes \mathbb{C})$  be the corresponding algebraic subtorus in  $H^1(X, \mathbb{C}^\times)$ . Then:

$$\begin{aligned} P \in \Omega_r^i(X) &\stackrel{(i)}{\iff} T \cap \mathcal{W}^i(X) \text{ is finite} \\ &\stackrel{(ii)}{\implies} \tau_1(T \cap \mathcal{W}^i(X)) = \{0\} \\ &\stackrel{(iii)}{\iff} (P \otimes \mathbb{C}) \cap \tau_1(\mathcal{W}^i(X)) = \{0\} \\ &\stackrel{(iv)}{\iff} P \cap L = \{0\}, \text{ for each } L \in \mathcal{C}_i(X), \end{aligned}$$

where in (i) we used Theorem 4.3, in (ii) we used Definition 5.1, in (iii) we used the fact that  $\tau_1$  commutes with intersections and  $\tau_1(T) = P \otimes \mathbb{C}$ , and in (iv) we used Definition 5.6.  $\square$

If  $\tau_1(\mathcal{W}^i(X)) = \{0\}$ , then the right-hand side of (34) is the whole Grassmannian, and the upper bound is tautological. Under appropriate hypothesis, though, the bound is sharp. The next corollary isolates a class of spaces for which this happens.

**Corollary 6.2.** *Suppose all positive-dimensional components of  $\mathcal{W}^i(X)$  are algebraic subtori. Then, for all  $r \geq 1$ , we have:*

$$(35) \quad \Omega_r^i(X) = \left( \bigcup_{L \in \mathcal{C}_i(X)} \{P \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid P \cap L \neq \{0\}\} \right)^c.$$

*Proof.* By hypothesis,  $\mathcal{W}^i(X) = W \cup Z$ , where  $W = \exp(\tau_1(\mathcal{W}^i(X)))$  is a finite union of algebraic subtori, and  $Z$  is a finite set. With notation as in the proof of Theorem 6.1, suppose  $\tau_1(T \cap \mathcal{W}^i(X)) = \{0\}$ . Then  $T \cap W = \{1\}$ , and thus  $T \cap \mathcal{W}^i(X)$  is finite. This shows implication (ii) can be reversed in this situation, thereby finishing the proof.  $\square$

A class of spaces for which the hypothesis of Corollary 6.2 holds are the “straight” spaces studied in [45]. In general, though, the inclusion from Theorem 6.1 is strict, as illustrated in the next example.

**Example 6.3.** Let  $L$  be the 2-component link denoted  $4_1^2$  in Rolfsen’s tables [37], and let  $X$  be its complement. Then  $\Delta_L = t_1 + t_2$ , and so the variety  $\mathcal{W}^1(X) \subset (\mathbb{C}^\times)^2$  consists of the identity 1, together with the 1-dimensional translated torus  $\{t_1 t_2^{-1} = -1\}$ . Using Theorem 4.4, we find that  $\Omega_2^1(X) = \emptyset$ . On the other hand,  $\tau_1(\mathcal{W}^1(X)) = \{0\}$ ; thus, the right-hand side of (34) equals  $\text{Gr}_2(\mathbb{Q}^2) = \{\text{pt}\}$ .

**6.2. The case of infinite cyclic covers.** When  $r = 1$ , the upper bound from Theorem 6.1 becomes an equality. To see why that is the case, set  $H = H^1(X, \mathbb{Q})$ , and identify the Grassmannian  $\text{Gr}_1(H)$  with the projective space  $\mathbb{Q}\mathbb{P}^{n-1} = \mathbb{P}(H)$ , where  $n = b_1(X)$ . Given a linear subspace  $L \subset H$ , denote its projectivization by  $\mathbb{P}(L)$ . We then have the following theorem, which recovers results from Dwyer–Fried [21] and Papadima–Suci [35]. For completeness, we include a proof adapted to our setting.

**Theorem 6.4.** *Let  $X$  be a CW-complex with finite  $k$ -skeleton. Set  $n = b_1(X)$ . Then  $\Omega_1^i(X) = \mathbb{Q}\mathbb{P}^{n-1} \setminus \bigcup_{L \in \mathcal{C}_i(X)} \mathbb{P}(L)$ , for all  $i \leq k$ ,*

*Proof.* Let  $P$  be a line in  $H = \mathbb{Q}^n$ . Set  $T = \exp(P \otimes \mathbb{C})$  and  $T' = T \cap \mathcal{W}^i(X)$ . Looking at the proof of Theorem 6.1, we only need to reverse implication (ii); that is, we need to show that  $\tau_1(T') = \{0\} \implies T'$  is finite. Now,  $T'$  is a Zariski closed subset of  $T \cong \mathbb{C}^\times$ . So, if  $T'$  were not finite, then  $T'$  would equal  $T$ , implying  $\tau_1(T') \cong \mathbb{C}$ .  $\square$

This theorem yields a nice qualitative result about the rank-1 Dwyer–Fried sets.

**Corollary 6.5** ([21]). *Each set  $\Omega_1^i(X)$  is the complement of a finite union of projective subspaces in  $\mathbb{Q}\mathbb{P}^{n-1}$ . In particular,  $\Omega_1^i(X)$  is a Zariski open set in  $\mathbb{Q}\mathbb{P}^{n-1}$ .*

Theorem 6.4 also gives an easy-to-use homological finiteness test for the regular, infinite cyclic covers of a space  $X$ .

**Corollary 6.6** ([35]). *Let  $X^\nu \rightarrow X$  be a connected, regular  $\mathbb{Z}$ -cover, classified by an epimorphism  $\nu: \pi_1(X) \twoheadrightarrow \mathbb{Z}$ , and let  $\bar{\nu} \in H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{C})$  be the corresponding cohomology class. Then  $\sum_{i=1}^k b_i(X^\nu) < \infty$  if and only if  $\bar{\nu} \notin \tau_1(\mathcal{W}^k(X))$ .*

**Example 6.7.** Let  $X$  be the link complement from Example 5.8. By Theorem 6.4, the set  $\Omega_1^1(X)$  is the complement in  $\mathbb{Q}\mathbb{P}^2$  of the points  $(0, 1, -1)$ ,  $(1, 0, -1)$ , and  $(1, -1, 0)$ . Thus, every regular  $\mathbb{Z}$ -cover  $X^\nu \rightarrow X$  has  $b_1(X^\nu) < \infty$ , except for the three covers specified by the those vectors.

## 7. SPECIAL SCHUBERT VARIETIES

In this section, we reinterpret the upper bound for the Dwyer–Fried sets in terms of a well-known geometric construction, by showing that each  $\Omega$ -invariant lies in the complement of an arrangement of (special) Schubert varieties.

**7.1. The incidence correspondence.** We start by recalling a classical construction from algebraic geometry (see [28, p. 69]). Fix a field  $\mathbb{k}$ , and let  $V$  be a homogeneous variety in  $\mathbb{k}^n$ . Consider the locus of  $r$ -planes in  $\mathbb{k}^n$  intersecting  $V$  non-trivially,

$$(36) \quad \sigma_r(V) = \{P \in \text{Gr}_r(\mathbb{k}^n) \mid P \cap V \neq \{0\}\}.$$

This set is a Zariski closed subset of the Grassmannian  $\text{Gr}_r(\mathbb{k}^n)$ , called the *variety of incident  $r$ -planes* to  $V$ . In particular,  $\sigma_r(\{0\}) = \emptyset$ .

Recall  $\dim \text{Gr}_r(\mathbb{k}^n) = r(n-r)$ . The following well-known fact (cf. [28, p. 153]) will be useful to us.

**Lemma 7.1.** *Let  $V$  be a homogeneous, irreducible variety in  $\mathbb{k}^n$ , of dimension  $m > 0$ . Then, for all  $0 < r < n-m$ , the incidence variety  $\sigma_r(V)$  is an irreducible subvariety of  $\text{Gr}_r(\mathbb{k}^n)$ , of dimension  $(r-1)(n-r) + m - 1$ .*

Particularly simple is the case when  $V$  is a linear subspace  $L \subset \mathbb{k}^n$ . The corresponding incidence variety,  $\sigma_r(L)$ , is known as the *special Schubert variety* defined by  $L$ . Clearly,  $\sigma_1(L) = \mathbb{P}(L)$ , viewed as a projective subspace in  $\mathbb{P}(\mathbb{k}^n)$ .

**Corollary 7.2.** *Let  $L$  be a non-zero, codimension  $d$  linear subspace in  $\mathbb{k}^n$ . Then  $\sigma_r(L)$  has codimension  $d - r + 1$  in  $\text{Gr}_r(\mathbb{k}^n)$ .*

To write down equations for the special Schubert varieties, start by embedding  $\text{Gr}_r(\mathbb{k}^n)$  in the projective space  $\mathbb{P}(\bigwedge^r \mathbb{k}^n)$  via the Plücker embedding, which sends an  $r$ -plane  $P$  to  $\bigwedge^r P$ , and let  $p_{i_1, \dots, i_r}(P)$ ,  $1 \leq i_1 < \dots < i_r \leq n$ , be the Plücker coordinates of this plane. Let  $L$  be an  $s$ -dimensional plane in  $\mathbb{k}^n$ , represented as the row space of an  $s \times n$  matrix. Then  $L$  meets an  $r$ -plane  $P$  non-trivially if and only if all the maximal minors of the matrix  $(\frac{L}{P})$  vanish. Laplace expansion of each minor along the rows of  $P$  yields a linear equation in the Plücker coordinates. We illustrate the procedure with an example.

**Example 7.3.** The Grassmannian  $\text{Gr}_2(\mathbb{k}^4)$  is the hypersurface in  $\mathbb{k}\mathbb{P}^5$  with equation

$$(37) \quad p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14} = 0.$$

Let  $L$  be a plane in  $\mathbb{k}^4$ . The corresponding Schubert variety,  $\sigma_2(L)$ , is the 3-fold in  $\text{Gr}_2(\mathbb{k}^4)$  cut out by the hyperplane

$$(38) \quad L_{34}p_{12} - L_{24}p_{13} + L_{14}p_{23} + L_{23}p_{14} - L_{13}p_{24} + L_{12}p_{34} = 0,$$

where  $L_{ij}$  is the  $(ij)$ -minor of a  $2 \times 4$  matrix representing  $L$ .

**7.2. Reinterpreting the upper bound for the  $\Omega$ -sets.** Using the notions discussed above, we may reformulate the exponential tangent cone upper bound from Theorem 6.1, as follows.

**Theorem 7.4.** *Let  $X$  be a CW-complex with finite  $k$ -skeleton. Then*

$$(39) \quad \Omega_r^i(X) \subseteq \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \bigcup_{L \in \mathcal{C}_i(X)} \sigma_r(L),$$

for all  $i \leq k$  and  $r \geq 1$ .

In other words, each Dwyer–Fried set  $\Omega_r^i(X)$  is contained in the complement of a Zariski closed subset of  $\text{Gr}_r(H^1(X, \mathbb{Q}))$ , namely, the union of special Schubert varieties corresponding to the subspaces belonging to the characteristic arrangement  $\mathcal{C}_i(X)$ .

**Remark 7.5.** If inclusion (39) holds as equality, then clearly  $\Omega_r^i(X)$  is a Zariski open subset of  $\text{Gr}_r(H^1(X, \mathbb{Q}))$ . Nevertheless, as we saw in Example 6.3, this inclusion can be strict even when the set  $\Omega_r^i(X)$  is open.

**Remark 7.6.** As shown in Theorem 6.4, inclusion (39) holds as equality if  $r = 1$ . On the other hand, if  $r > 1$ , the sets  $\Omega_r^i(X)$  are not necessarily open. This rather surprising fact was first noticed by Dwyer and Fried, who constructed in [21] a 3-dimensional cell complex of the form  $X = (T^3 \vee S^2) \cup_{\varphi} e^3$  for which  $\Omega_2^2(X)$  is a finite set (see also [44] for more details). We will come back to this topic in §8, where we will give an example of a finitely presented group  $G$  for which  $\Omega_2^1(G)$  is a single point.

Corollary 6.2 can also be reformulated, as follows.

**Corollary 7.7.** *Suppose all positive-dimensional components of  $\mathcal{W}^i(X)$  are algebraic subtori. Then, for all  $r \geq 1$ , we have:*

$$(40) \quad \Omega_r^i(X) = \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X))).$$

In particular,  $\Omega_r^i(X)$  is a Zariski open subset in  $\text{Gr}_r(H^1(X, \mathbb{Q}))$ .

**Example 7.8.** Let  $G = F_2 \times F_2$ . By Corollary 2.10, the first characteristic variety  $\mathcal{V}^1(G)$  consists of two, 2-dimensional algebraic subtori in  $(\mathbb{C}^\times)^4$ , namely  $T_1 = \{t_1 = t_2 = 1\}$  and  $T_2 = \{t_3 = t_4 = 1\}$ . Thus, the characteristic subspace arrangement  $\mathcal{C}_1(G)$  consists of two planes in  $\mathbb{Q}^4$ , namely,  $L_1 = \{x_1 = x_2 = 0\}$  and  $L_2 = \{x_3 = x_4 = 0\}$ .

Now identify the Grassmannian  $\text{Gr}_2(\mathbb{Q}^4)$  with the quadric in  $\mathbb{Q}\mathbb{P}^5$  given by equation (37). In view of formulas (40) and (38), then, the set  $\Omega_2^1(X)$  is the complement in  $\text{Gr}_2(\mathbb{Q}^4)$  of the variety cut out by the hyperplanes  $\{p_{12} = 0\}$  and  $\{p_{34} = 0\}$ .

In favorable situations, the information provided by Theorem 7.4 allows us to identify precisely the  $\Omega$ -sets.

**Corollary 7.9.** *Suppose the characteristic arrangement  $\mathcal{C}_i(X)$  contains a non-zero subspace of codimension  $d$ . Then  $\Omega_r^i(X) = \emptyset$ , for all  $r > d$ .*

*Proof.* Let  $L$  be a subspace in  $\mathcal{C}_i(X)$ , with  $L \neq \{0\}$  and  $\text{codim}(L) = d$ . Suppose  $r \geq d + 1$ . By Corollary 7.2, we have  $\text{codim}(\sigma_r(L)) = d - r + 1 \leq 0$ , and so  $\sigma_r(L) = \text{Gr}_r(H^1(X, \mathbb{Q}))$ . In view of (39), this forces  $\Omega_r^i(X)$  to be empty.  $\square$

**Corollary 7.10.** *Let  $X^\alpha$  be the maximal free abelian cover of  $X$ . If  $\tau_1(\mathcal{W}^1(X)) \neq \{0\}$ , then  $b_1(X^\alpha) = \infty$ .*

*Proof.* Set  $n = b_1(X)$ . By hypothesis, the arrangement  $\mathcal{C}_1(X)$  contains a non-zero subspace, say  $L$ . Since  $\text{codim}(L) \leq n - 1$ , the previous corollary implies  $\Omega_n^1(X) = \emptyset$ . The desired conclusion follows from (25).  $\square$

The converse to the implication from Corollary 7.10 does not hold. For instance, if  $X$  is the complement to the 2-component link from Example 6.3, then  $\tau_1(\mathcal{W}^1(X)) = \{0\}$ , yet  $\Omega_2^1(X) = \emptyset$ , and so  $b_1(X^\alpha) = \infty$ .

## 8. TRANSLATED TORI

In this section, we give a fairly complete description of the Dwyer–Fried sets  $\Omega_r^i(X)$  in the case when all the positive-dimensional components of the characteristic variety  $\mathcal{W}^i(X)$  are torsion-translated subtori of the character torus of  $\pi_1(X)$ .

**8.1. Intersections of translated tori.** We start by recalling some basic terminology. Fix a complex algebraic torus  $(\mathbb{C}^\times)^n$ . A *subtorus* is a connected algebraic subgroup; every such subgroup is of the form  $T = \exp(L \otimes \mathbb{C})$ , for some linear subspace  $L \subset \mathbb{Q}^n$ . A *translated subtorus* is a coset of a subtorus, i.e, a subvariety of  $(\mathbb{C}^\times)^n$  of the form  $\rho \cdot T$ ; if  $\rho \in (\mathbb{C}^\times)^n$  can be chosen to have finite order, we say  $\rho \cdot T$  is a *torsion-translated subtorus*.

For our purposes, it is important to understand the way translated subtori intersect inside an algebraic torus. Much of the relevant theory was worked out by E. Hironaka in [29]; the theory was further developed by Suci, Yang, and Zhao in [47]. The specific result we shall need can be formulated as follows.

**Proposition 8.1** ([47]). *Let  $T_1 = \exp(L_1 \otimes \mathbb{C})$  and  $T_2 = \exp(L_2 \otimes \mathbb{C})$  be two algebraic subtori in  $(\mathbb{C}^\times)^n$ , and let  $\rho_1$  and  $\rho_2$  be two elements in  $(\mathbb{C}^\times)^n$ . Then*

- (1) *The variety  $Q = \rho_1 T_1 \cap \rho_2 T_2$  is non-empty if and only if  $\rho_1 \rho_2^{-1}$  belongs to the subgroup  $T_1 \cdot T_2 = \exp((L_1 + L_2) \otimes \mathbb{C})$ .*
- (2) *If the above condition is satisfied, then  $\dim Q = \dim(T_1 \cap T_2)$ .*

Given a linear subspace  $L \subset \mathbb{Q}^n$  and an element  $\rho \in (\mathbb{C}^\times)^n$ , it is convenient to introduce the following notation:

$$(41) \quad \sigma_r(L, \rho) = \{P \in \text{Gr}_r(\mathbb{Q}^n) \mid \rho \in \exp((P + L) \otimes \mathbb{C}) \text{ and } P \cap L \neq \{0\}\}.$$

Clearly,  $\sigma_r(L, \rho) \subseteq \sigma_r(L)$  for all  $\rho$ , with equality if  $\rho \in \exp(L \otimes \mathbb{C})$ . If  $\dim L = 1$ , then  $\sigma_r(L, \rho)$  consists of those  $r$ -planes  $P$  which contain the line  $L$  and for which  $\rho \in \exp(P \otimes \mathbb{C})$ ; in particular, if  $\rho \notin \exp(L \otimes \mathbb{C})$ , then  $\sigma_1(L, \rho) = \emptyset$ .

**Remark 8.2.** Unlike the special Schubert varieties  $\sigma_r(L)$ , the sets  $\sigma_r(L, \rho)$  need not be Zariski closed subsets of the rational Grassmannian. For instance, if  $L$  has dimension  $2 \leq d \leq n - 1$ , and  $\rho$  is an element of finite-order in  $(\mathbb{C}^\times)^n \setminus \exp(L \otimes \mathbb{C})$ , then, as we shall see in the proof of Theorem 8.5, the set  $\sigma_r(L, \rho)$  is not a closed subset of  $\text{Gr}_r(\mathbb{Q}^n)$ , for any  $2 \leq r \leq d$ . In fact, as we shall see in the proof of Proposition 8.6, if  $d = n - 1$ , then  $\sigma_r(L, \rho) = \text{Gr}_r(\mathbb{Q}^n) \setminus \text{Gr}_r(L)$ .

Proposition 8.1 has an immediate corollary.

**Corollary 8.3.** *Let  $\rho T$  be a translated subtorus in  $(\mathbb{C}^\times)^n$ . Write  $T = \exp(L \otimes \mathbb{C})$ , for some linear subspace  $L \subset \mathbb{Q}^n$ . Given a subspace  $P \in \text{Gr}_r(\mathbb{Q}^n)$ , we have*

$$(42) \quad \dim_{\mathbb{C}}(\exp(P \otimes \mathbb{C}) \cap \rho T) > 0 \iff P \in \sigma_r(L, \rho).$$



**8.2. Translated tori and Dwyer–Fried sets.** We are now ready to state and prove the main result of this section.

**Theorem 8.4.** *Let  $X$  be a CW-complex with finite  $k$ -skeleton, for some  $k \geq 1$ . For a fixed integer  $i \leq k$ , suppose all positive-dimensional components of  $\mathcal{W}^i(X)$  are translated subtori in  $H^1(X, \mathbb{C}^\times)^\circ$ . Write*

$$(43) \quad \mathcal{W}^i(X) = \left( \bigcup_{\alpha} \rho_{\alpha} T_{\alpha} \right) \cup Z,$$

where  $Z$  is a finite set and  $T_{\alpha} = \exp(L_{\alpha} \otimes \mathbb{C})$ , for some linear subspace  $L_{\alpha} \subset H^1(X, \mathbb{Q})$  and some  $\rho_{\alpha} \in H^1(X, \mathbb{C}^\times)^\circ$ . Then, for all  $r \geq 1$ ,

$$(44) \quad \Omega_r^i(X) = \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \bigcup_{\alpha} \sigma_r(L_{\alpha}, \rho_{\alpha}).$$

*Proof.* Fix an  $r$ -plane  $P$  inside  $H^1(X, \mathbb{Q})$ , and let  $T = \exp(P \otimes \mathbb{C})$  be the corresponding algebraic subtorus in  $H^1(X, \mathbb{C}^\times)^\circ$ . Then:

$$\begin{aligned} P \in \Omega_r^i(X) &\iff T \cap \mathcal{W}^i(X) \text{ is finite,} && \text{by Theorem 4.3} \\ &\iff T \cap \rho_{\alpha} T_{\alpha} \text{ is finite, for all } \alpha, && \text{by assumption (43)} \\ &\iff P \notin \sigma_r(L_{\alpha}, \rho_{\alpha}), \text{ for all } \alpha, && \text{by Corollary 8.3.} \end{aligned}$$

This ends the proof.  $\square$

If all the translation factors  $\rho_{\alpha}$  in (43) are equal to 1, then we are in the situation of Corollary 7.7, and the Dwyer–Fried sets  $\Omega_r^i(X)$  are Zariski open subsets of  $\text{Gr}_r(H^1(X, \mathbb{Q}))$ . On the other hand, if one of those translation factors is a non-trivial root of unity, the corresponding subtorus has dimension  $r > 1$ , and that subtorus intersects the other subtori transversely at the identity, then, as we show next, the set  $\Omega_r^i(X)$  is not open, even for the usual topology on the Grassmannian.

**Theorem 8.5.** *As above, suppose all positive-dimensional components of  $\mathcal{W}^i(X)$  are translated subtori in  $H^1(X, \mathbb{C}^\times)^\circ$ , of the form  $\rho_{\alpha} T_{\alpha}$ . Furthermore, suppose there is a subtorus  $T$  of dimension  $d \geq 2$  such that:*

- (1) *There is at least one component  $\rho_{\beta} T_{\beta}$  with  $T_{\beta} = T$ .*
- (2) *If  $T_{\alpha} = T$ , then  $\rho_{\alpha}$  has finite order (modulo  $T$ ) and  $\rho_{\alpha} \notin T$ .*
- (3) *If  $T_{\alpha} \neq T$ , then  $\tau_1(T_{\alpha}) \cap \tau_1(T) = \{0\}$ .*

*Then, for each  $2 \leq r \leq d$ , the set  $\Omega_r^i(X)$  is not open in  $\text{Gr}_r(H^1(X, \mathbb{Q}))$ .*

*Proof.* Set  $n = b_1(X)$ , and identify  $H^1(X, \mathbb{Q}) = \mathbb{Q}^n$ . Write  $T_{\alpha} = \exp(L_{\alpha} \otimes \mathbb{C})$ , with  $L_{\alpha} = \tau_1(T_{\alpha})$  a linear subspace of  $\mathbb{Q}^n$ , and similarly,  $T = \exp(L \otimes \mathbb{C})$ . For each component  $\rho_{\alpha} T_{\alpha}$  with  $L_{\alpha} = L$ , write  $\rho_{\alpha} = \exp(2\pi i \lambda_{\alpha})$ ; by assumption (2), then, we have  $\lambda_{\alpha} \in \mathbb{Q}^n$  but  $\lambda_{\alpha} \notin L_{\alpha} + \mathbb{Z}^n$ .

Fix a basis  $\{v_1, \dots, v_d\}$  for  $L$  and an integer  $2 \leq r \leq d$ . By assumption (1), there is an index  $\beta$  for which  $L_{\beta} = L$ . Consider the  $r$ -dimensional subspaces

$$P = \text{span}\{v_1, \dots, v_r\} \quad \text{and} \quad P_q = \text{span}\{v_1, \dots, v_{r-1}, v_r + \frac{\lambda_{\beta}}{q}\}.$$

Clearly,  $P_q \rightarrow P$ . By Theorem 8.4, we have that  $\Omega_r^i(X) = \text{Gr}_r(\mathbb{Q}^n) \setminus \bigcup_{\alpha} \sigma_r(L_{\alpha}, \rho_{\alpha})$ . Thus, to finish the proof, it is enough to show that  $P \notin \sigma_r(L_{\alpha}, \rho_{\alpha})$ , for all  $\alpha$ , yet  $P_q \in \sigma_r(L_{\beta}, \rho_{\beta})$ , for all  $q$ .

To prove the first claim, first note that  $P \subseteq L$ . Next, consider an index  $\alpha$  so that  $L_{\alpha} \neq L$ . By assumption (3), we have that  $L_{\alpha} \cap L = \{0\}$ ; thus,  $P \notin \sigma_r(L_{\alpha}, \rho_{\alpha})$ , and so

$P \notin \sigma_r(L_\alpha, \rho_\alpha)$ . Finally, consider an index  $\alpha$  so that  $L_\alpha = L$ . Then  $\rho_\alpha \notin \exp((P + L_\alpha) \otimes \mathbb{C}) = T_\alpha$ ; hence,  $P \notin \sigma_r(L_\alpha, \rho_\alpha)$ .

To prove the second claim, note that  $\dim(P_q \cap L_\beta) = r - 1 > 0$ . Furthermore, note that

$$\rho_\beta = \exp(-2\pi i q v_r) \cdot \exp(2\pi i q (v_r + \frac{\lambda_\beta}{q})) \in \exp(L_\beta \otimes \mathbb{C}) \cdot \exp(P_q \otimes \mathbb{C}).$$

Corollary 8.3 now implies that  $P_q \in \sigma_r(L_\beta, \rho_\beta)$ , and we are done.  $\square$

In particular, if the only positive-dimensional component of  $\mathcal{W}^i(X)$  is a torsion-translated subtorus of dimension  $d \geq 2$ , then the sets  $\Omega_2^i(X), \dots, \Omega_d^i(X)$  are not open.

**8.3. Examples of non-open  $\Omega$ -invariants.** We now isolate a situation where we can explicitly identify certain Dwyer–Fried sets which are not open subsets of the rational Grassmannian.

**Proposition 8.6.** *Let  $X$  be a CW-complex with finite  $k$ -skeleton, and set  $n = b_1(X)$ . Suppose that, for some  $i \leq k$ , there is an  $(n - 1)$ -dimensional subspace  $L \subset H^1(X, \mathbb{Q})$  such that  $\mathcal{W}^i(X) = (\bigcup_\alpha \rho_\alpha T) \cup Z$ , where  $Z$  is a finite set,  $T = \exp(L \otimes \mathbb{C})$  and  $\rho_\alpha \notin T$ , for all  $\alpha$ . Then*

$$\Omega_r^i(X) = \begin{cases} \mathbb{Q}\mathbb{P}^{n-1} & \text{if } r = 1, \\ \text{Gr}_r(L) & \text{if } 1 < r < n, \\ \emptyset & \text{if } r \geq n. \end{cases}$$

*In particular,  $\Omega_r^i(X)$  is not open for  $1 < r < n$ , and  $\Omega_{n-1}^i(X) = \{L\}$ .*

*Proof.* First note that  $\tau_1(\mathcal{W}^i(X)) = \{0\}$ , since  $1 \notin \rho_\alpha T$ ; thus, by Theorem 6.4,  $\Omega_1^i(X) = \mathbb{Q}\mathbb{P}^{n-1}$ . The fact that  $\Omega_n^i(X) = \emptyset$  follows from Theorem 4.4.

Now assume  $2 \leq r \leq n - 1$ , and suppose  $P$  is an  $r$ -plane in  $\mathbb{Q}^n$ . If  $P \subsetneq L$ , then  $P + L = \mathbb{Q}^n$  and  $\dim(P \cap L) \geq r - 1 > 0$ ; thus,  $P \in \sigma_r(L, \rho_\alpha)$ . Conversely, if  $P \subseteq L$ , then clearly  $P \notin \sigma_r(L, \rho_\alpha)$ . The desired conclusion follows from Theorem 8.4.  $\square$

We illustrate this proposition with a concrete example—as far as we know, the first of its kind—of a finitely presented group  $G$  for which the set  $\Omega_2^1(G)$  is not open.

**Example 8.7.** Consider the group  $G$  with generators  $x_1, x_2, x_3$  and relators  $r_1 = [x_1^2, x_2]$ ,  $r_2 = [x_1, x_3]$ ,  $r_3 = x_1[x_2, x_3]x_1^{-1}[x_2, x_3]$ . Note that  $G_{\text{ab}} = \mathbb{Z}^3$ . Computing Fox derivatives, we find that

$$\partial_2^{\text{ab}} = \begin{pmatrix} (x_2 - 1)(1 + x_1) & (1 - x_1)(1 + x_1) & 0 \\ x_3 - 1 & 0 & 1 - x_1 \\ 0 & (x_3 - 1)(1 + x_1) & (1 - x_2)(1 + x_1) \end{pmatrix}.$$

Identifying  $\widehat{G} = (\mathbb{C}^\times)^3$ , we obtain  $\mathcal{W}^1(G) = \{1\} \cup \{t \in (\mathbb{C}^\times)^3 \mid t_1 = -1\}$ . By Proposition 8.6, the set  $\Omega_2^1(G)$  consists of a single point in  $\text{Gr}_2(\mathbb{Q}^3)$ , corresponding to the plane  $x_1 = 0$ . In particular,  $\Omega_2^1(G)$  is not open, not even in the usual topology on  $\mathbb{Q}\mathbb{P}^2$ .

## 9. KÄHLER MANIFOLDS

In this section, we discuss the characteristic varieties (also known in this context as the Green–Lazarsfeld sets) and the Dwyer–Fried sets of compact Kähler manifolds, highlighting the manner in which orbifold fibrations determine these sets in degree 1.

**9.1. Maps to 2-orbifolds.** Let  $\Sigma_g$  be a Riemann surface of genus  $g \geq 1$ . As we saw in Example 2.7, the characteristic varieties of  $\Sigma_g$  either fill the whole character torus  $H^1(\Sigma_g, \mathbb{C}^\times) = (\mathbb{C}^\times)^{2g}$ , or consist only of the identity. In particular, if  $g > 1$ , then  $\mathcal{V}^1(\Sigma_g) = (\mathbb{C}^\times)^{2g}$ .

Following the approach of Dimca [17], Delzant [16], Campana [12], and Artal Bartolo, Cogolludo, and Matei [4], let us consider the more general situation of 2-orbifolds. Fix points  $q_1, \dots, q_t$  in  $\Sigma_g$ , and assign to these points integer weights  $m_1, \dots, m_t$  with  $m_i \geq 2$ . The orbifold fundamental group  $\Gamma = \pi_1^{\text{orb}}(\Sigma_g, \mathbf{m})$  associated to these data may be presented as

$$(45) \quad \Gamma = \left\langle \begin{array}{c} x_1, \dots, x_g, y_1, \dots, y_g, \\ z_1, \dots, z_t \end{array} \mid \begin{array}{l} [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_t = 1, \\ z_1^{m_1} = \cdots = z_t^{m_t} = 1 \end{array} \right\rangle.$$

Clearly, we have an epimorphism  $\kappa: \Gamma \rightarrow \pi_1(\Sigma_g)$ , obtained by sending  $z_i \mapsto 1$ . Upon abelianizing, we obtain an isomorphism  $\Gamma_{\text{ab}} \cong \pi_1(\Sigma_g)_{\text{ab}} \oplus A$ , where

$$(46) \quad A := \text{Tors}(\Gamma_{\text{ab}}) = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t} / (1, \dots, 1).$$

Evidently, the group  $A$  has order  $m_1 \cdots m_t / \text{lcm}(m_1, \dots, m_t)$ ; in particular, if  $t = 0$  or  $t = 1$ , then  $A = 0$ , but otherwise,  $A \neq 0$ .

Identify  $\widehat{\Gamma} = \widehat{\Gamma}^\circ \times \widehat{A}$ , where  $\widehat{\Gamma}^\circ = \widehat{\kappa}(\widehat{\pi_1(\Sigma_g)}) \cong (\mathbb{C}^\times)^{2g}$ . A Fox calculus computation as in [4, Proposition 3.11] shows that

$$(47) \quad \mathcal{V}^1(\Gamma) = \begin{cases} \widehat{\Gamma} & \text{if } g \geq 2, \\ (\widehat{\Gamma} \setminus \widehat{\Gamma}^\circ) \cup \{1\} & \text{if } g = 1 \text{ and } t > 1, \\ \{1\} & \text{if } g = 1 \text{ and } t \leq 1. \end{cases}$$

**Proposition 9.1.** *Let  $G$  be a finitely generated group, and suppose there is an epimorphism  $\varphi: G \twoheadrightarrow \Gamma$ , where  $\Gamma = \pi_1^{\text{orb}}(\Sigma_g, (m_1, \dots, m_t))$ , with either  $g \geq 2$ , or  $g = 1$  and  $t \geq 2$ . Set  $A = \text{Tors}(\Gamma_{\text{ab}})$ . Then*

$$(48) \quad \mathcal{V}^1(G) \supseteq \bigcup_{\rho} \rho \cdot T,$$

where  $T$  is the  $(2g)$ -dimensional subtorus of  $\widehat{G}$  obtained by pulling back  $\widehat{\pi_1(\Sigma_g)}$  along the map  $\varphi_0 = \kappa \circ \varphi: G \twoheadrightarrow \pi_1(\Sigma_g)$ , and the union is taken over the set  $\widehat{\varphi}(\widehat{A})$  in the first case, and  $\widehat{\varphi}(\widehat{A}) \setminus \{1\}$  in the second case.

*Proof.* By Lemma 2.13, the induced morphism on character groups,  $\widehat{\varphi}: \widehat{\Gamma} \hookrightarrow \widehat{G}$ , embeds the characteristic variety  $\mathcal{V}^1(\Gamma)$  into  $\mathcal{V}^1(G)$ . Applying formula (47) and the discussion preceding it ends the proof.  $\square$

Note that, in either case, the right side of (48) is a finite union of torsion-translated subtori of  $\widehat{G}$ .

**9.2. The Green–Lazarsfeld sets.** Let  $M$  be a compact, connected, Kähler manifold, for instance, a smooth, complex projective variety. The structure of the characteristic varieties of such manifolds was determined by Green and Lazarsfeld in [25, 26], building on work of Castelnuovo and de Franchis, Beauville [6], and Catanese [13]. For this reason, the varieties  $\mathcal{V}_d^i(M)$  are also known in this context as the *Green–Lazarsfeld sets* of  $M$ .

The theory was further amplified by Simpson [39], Ein and Lazarsfeld [22], Arapura [3], and Campana [11], with some of the latest developments appearing in [17, 16, 20, 12, 4]. The relationship between the present definition of characteristic varieties and the original

definition from [25] is discussed at length by Budur in [10]. The basic nature of the Green–Lazarsfeld sets is summarized in the following theorem.

**Theorem 9.2** ([26, 6, 39, 3]). *Each characteristic variety  $\mathcal{V}_d^i(M)$  of a compact Kähler manifold  $M$  is a finite union of unitary translates of algebraic subtori of  $H^1(M, \mathbb{C}^\times)$ . Furthermore, if  $M$  is projective, then all the translates are by torsion characters.*

In degree  $i = 1$ , the structure of the Green–Lazarsfeld set  $\mathcal{V}^1(M) = \mathcal{V}_1^1(M) \cup \{1\}$  can be made more precise. First, we need some background on orbifold fibrations (also known as orbifold morphisms, or pencils).

As before, let  $\Sigma_g$  be a Riemann surface of genus  $g \geq 1$ , with marked points  $q_1, \dots, q_t$ , and weight vector  $\mathbf{m} = (m_1, \dots, m_t)$ , where  $m_i \geq 2$  and  $|\mathbf{m}| := t \geq 0$ . A surjective map  $f: M \rightarrow (\Sigma_g, \mathbf{m})$  is called an *orbifold fibration* if  $f$  is holomorphic, the fiber over any non-marked point is connected, and, for every point  $q_i$ , the multiplicity of the fiber  $f^{-1}(q_i)$  equals  $m_i$ . Such a map induces an epimorphism  $f_\#: \pi_1(M) \twoheadrightarrow \Gamma$ , where  $\Gamma = \pi_1^{\text{orb}}(\Sigma_g, \mathbf{m})$  is the orbifold fundamental group described in (45). By Proposition 9.1, the induced morphism of character groups,  $\widehat{f_\#}: \widehat{\Gamma} \hookrightarrow \widehat{\pi_1(M)}$ , sends  $\mathcal{V}^1(\Gamma)$  to a union of (possibly torsion-translated) subtori inside  $\mathcal{V}^1(M)$ .

Two orbifold fibrations,  $f: M \rightarrow (\Sigma_g, \mathbf{m})$  and  $f': M \rightarrow (\Sigma_{g'}, \mathbf{m}')$ , are equivalent if there is a biholomorphic map  $h: \Sigma_g \rightarrow \Sigma_{g'}$  which sends marked points to marked points, while preserving multiplicities. Write  $\chi^{\text{orb}}(\Sigma_g, \mathbf{m}) = 2 - 2g - \sum_{i=1}^t (1 - 1/m_i)$ . As shown by Delzant [16], a Kähler manifold  $M$  admits only finitely many equivalence classes of orbifold fibrations for which the orbifold Euler characteristic of the base is negative.

The next theorem, which is a distillation of several results from the quoted sources, shows that *all* positive-dimensional components in the first characteristic variety of  $M$  arise by pullback along this finite set of pencils.

**Theorem 9.3** ([6, 3, 11, 17, 16, 12, 4]). *Let  $M$  be a compact Kähler manifold. Then*

$$(49) \quad \mathcal{V}^1(M) = \bigcup_{\alpha} \widehat{(f_\alpha)_\#}(\mathcal{V}^1(\pi_1^{\text{orb}}(\Sigma_{g_\alpha}, \mathbf{m}_\alpha))) \cup Z,$$

where  $Z$  is a finite set of torsion characters, and the union runs over the (finite) set of equivalence classes of orbifold fibrations  $f_\alpha: M \rightarrow (\Sigma_{g_\alpha}, \mathbf{m}_\alpha)$  with either  $g_\alpha \geq 2$ , or  $g_\alpha = 1$  and  $|\mathbf{m}_\alpha| \geq 2$ .

In particular, every positive-dimensional component of  $\mathcal{V}^1(M)$  is of the form  $\rho \cdot T$ , with  $T$  a  $(2g)$ -dimensional algebraic subtorus in  $H^1(M, \mathbb{C}^\times)$ , and  $\rho$  of finite order (modulo  $T$ ). Clearly, if  $\rho \in T$ , then  $g \geq 2$ . Moreover, if  $\rho \notin T$  and  $g \geq 2$ , then  $T$  is also a component of  $\mathcal{V}^1(M)$ , but if  $g = 1$ , then  $T$  is *not* a component of  $\mathcal{V}^1(M)$ . Let us record a particularly simple situation as a corollary.

**Corollary 9.4.** *Let  $M$  be a compact Kähler manifold, and suppose  $M$  admits no orbifold fibrations with multiple fibers. Then*

$$(50) \quad \mathcal{V}^1(M) = \bigcup_{\alpha} f_\alpha^*(H^1(\Sigma_{g_\alpha}, \mathbb{C}^\times)) \cup Z,$$

where  $Z$  is a finite set of torsion characters, and the union runs over the set of equivalence classes of orbifold fibrations  $f_\alpha: M \rightarrow \Sigma_{g_\alpha}$  with  $g_\alpha \geq 2$ , and  $f_\alpha^*: H^1(\Sigma_{g_\alpha}, \mathbb{C}^\times) \hookrightarrow H^1(M, \mathbb{C}^\times)$  is the induced homomorphism in cohomology.

Another simple situation is the one considered by Green and Lazarsfeld [25] and Hacon and Pardini [27] in a slightly less general setting.

**Proposition 9.5.** *Let  $M$  be a compact Kähler manifold, and suppose there is no orbifold fibration  $M \rightarrow \Sigma_g$  with  $g \geq 2$ . Then*

- (1)  $\dim \mathcal{V}^1(M) = 0$ .
- (2) *If, moreover,  $p_g(M) = q(M) = 3$ , then  $\mathcal{V}^1(M) = \{1\}$ .*

*Proof.* Part (1) follows directly from Theorem 9.3 and the discussion following it.

Part (2) follows from the proof of [27, Proposition 2.8], which crucially relies on Theorem 1.2 (1.2.3) from [22].  $\square$

**Example 9.6.** Following [14], consider the symmetric product  $M = (\Sigma_3 \times \Sigma_3)/\sigma$ , where  $\sigma$  is the involution interchanging the two factors. Then  $M$  is a minimal surface of general type with  $p_g(M) = q(M) = 3$  and  $K_M^2 = 6$ . As noted in [27], this surface has no irrational pencils of genus  $g \geq 2$ , and thus  $\mathcal{V}^1(M) = \{1\}$ .

**9.3. Characteristic subspace arrangements.** As an immediate corollary to Theorem 9.3, we also obtain the following characterization of the exponential tangent cone to the first Green–Lazarsfeld set of  $M$ .

**Corollary 9.7.** *Let  $M$  be a compact Kähler manifold. Then*

$$(51) \quad \tau_1(\mathcal{W}^1(M)) = \bigcup_{\alpha} f_{\alpha}^*(H^1(\Sigma_{g_{\alpha}}, \mathbb{C})),$$

where the union runs over the set of equivalence classes of orbifold fibrations  $f_{\alpha}: M \rightarrow \Sigma_{g_{\alpha}}$  with  $g_{\alpha} \geq 2$ , and  $f_{\alpha}^*: H^1(\Sigma_{g_{\alpha}}, \mathbb{C}) \hookrightarrow H^1(M, \mathbb{C})$  is the induced homomorphism in cohomology.

Put differently, the characteristic arrangement  $\mathcal{C}_1(M)$  consists of the linear subspaces  $f_{\alpha}^*(H^1(\Sigma_{g_{\alpha}}, \mathbb{Q})) \subseteq H^1(M, \mathbb{Q})$  arising by pullback along orbifold fibrations with base genus  $g_{\alpha} \geq 2$ . Note that each of these subspaces has dimension  $2g_{\alpha} \geq 4$ .

It follows from [22, 20] that  $\tau_1(\mathcal{W}^1(M))$  coincides with the resonance variety  $\mathcal{R}^1(M, \mathbb{C})$  defined in (10); thus, the arrangement  $\mathcal{C}_1(M)$  depends only on the cup-product map  $H^1(M, \mathbb{Q}) \wedge H^1(M, \mathbb{Q}) \rightarrow H^2(M, \mathbb{Q})$ . In the projective setting, a bit more can be said.

**Theorem 9.8** ([18]). *Let  $M$  be a smooth, complex projective variety, and let  $\rho_{\alpha}T_{\alpha}$  and  $\rho_{\beta}T_{\beta}$  be two distinct components of  $\mathcal{V}^1(M)$ . Then either  $T_{\alpha} = T_{\beta}$ , or  $\tau_1(T_{\alpha}) \cap \tau_1(T_{\beta}) = \{0\}$ . Furthermore,  $\rho_{\alpha}T_{\alpha} \cap \rho_{\beta}T_{\beta}$  is a finite set of torsion characters.*

Therefore, the linear subspaces  $\tau_1(T_{\alpha}) = f_{\alpha}^*(H^1(\Sigma_{g_{\alpha}}, \mathbb{Q}))$  comprising the arrangement  $\mathcal{C}_1(M)$  intersect pairwise transversely.

**9.4. Dwyer–Fried sets.** The above results of the Green–Lazarsfeld sets of compact Kähler manifolds  $M$  can be used to describe the Dwyer–Fried invariants of such manifolds. For instance, if  $\mathcal{W}^i(M)$  contains no positive-dimensional translated subtori, then Corollary 7.7 and Theorem 9.2 insure that  $\Omega_r^i(M) = \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(M)))^{\circ}$ , for all  $r \geq 1$ . When  $i = 1$ , we can be much more concrete.

Given an orbifold fibration  $f: M \rightarrow (\Sigma_g, \mathbf{m})$ , write  $L = f^*(H^1(\Sigma_g, \mathbb{Q}))$  and  $A = \text{Tors}(\pi_1^{\text{orb}}(\Sigma_g, \mathbf{m}))$ . Furthermore, denote by  $\varphi: \pi_1(M) \twoheadrightarrow \pi_1^{\text{orb}}(\Sigma_g, \mathbf{m})$  and  $\varphi_0: \pi_1(M) \rightarrow \pi_1(\Sigma_g)$  the homomorphisms induced by  $f$ , and identify  $\widehat{\varphi_0(\pi_1(\Sigma_g))} = f^*(H^1(\Sigma_g, \mathbb{C}^{\times}))$  with the subtorus  $T = \exp(L \otimes \mathbb{C})$ .

**Theorem 9.9.** *Let  $M$  be a compact Kähler manifold. For all  $r \geq 1$ ,*

$$(52) \quad \Omega_r^1(M) = \text{Gr}_r(H^1(M, \mathbb{Q})) \setminus \bigcup_{\alpha} \bigcup_{\rho} \sigma_r(L_{\alpha}, \rho),$$

where the first union runs over the set of equivalence classes of orbifold fibrations  $f_\alpha: M \rightarrow (\Sigma_{g_\alpha}, \mathbf{m}_\alpha)$  with either  $g_\alpha \geq 2$ , or  $g_\alpha = 1$  and  $|\mathbf{m}_\alpha| \geq 2$ , while the second union runs over  $\rho \in \hat{\varphi}_\alpha(\hat{A}_\alpha)$  and  $\rho \in \hat{\varphi}_\alpha(\hat{A}_\alpha) \setminus \{1\}$ , respectively.

*Proof.* Follows from Theorem 8.4 on one hand, and Proposition 9.1 and Theorem 9.3 on the other hand.  $\square$

Next, we give a cohomological upper bound for the Dwyer–Fried sets  $\Omega_r^1(M)$ , and single out a couple of situations when that bound is attained.

**Theorem 9.10.** *Let  $M$  be a compact Kähler manifold. Then, for all  $r \geq 1$ ,*

$$(53) \quad \Omega_r^1(M) \subseteq \text{Gr}_r(H^1(M, \mathbb{Q})) \setminus \bigcup_{\alpha} \sigma_r(f_\alpha^*(H^1(\Sigma_{g_\alpha}, \mathbb{Q}))),$$

where the union runs over the set of equivalence classes of orbifold fibrations  $f_\alpha: M \rightarrow \Sigma_{g_\alpha}$  with  $g_\alpha \geq 2$ . Moreover, inclusion (53) holds as equality, provided either

- (1)  $r = 1$ , or
- (2)  $M$  admits no orbifold fibrations with multiple fibers.

*Proof.* Formula (53) is a direct consequence of Theorem 7.4 and Corollary 9.7.

Claim (1) now follows from Theorem 6.4, while Claim (2) follows from Corollaries 7.7 and 9.4.  $\square$

In the situations (1) and (2) recorded above, each Dwyer–Fried set  $\Omega_r^1(M)$  is the complement of a finite union of special Schubert varieties; in particular, an open subset of  $\text{Gr}_r(H^1(M, \mathbb{Q}))$ . Other instances when this happens are recorded in the next proposition.

**Proposition 9.11.** *Let  $M$  be a compact Kähler manifold.*

- (1) *If  $M$  admits an orbifold fibration with base genus  $g \geq 2$ , then  $\Omega_r^1(M) = \emptyset$ , for all  $r > b_1(M) - 2g$ .*
- (2) *If  $M$  does not admit an orbifold fibration with base genus  $g \geq 2$ , then  $\Omega_r^1(M) = \text{Gr}_r(H^1(M, \mathbb{Q}))$ , for all  $r \geq 1$ .*

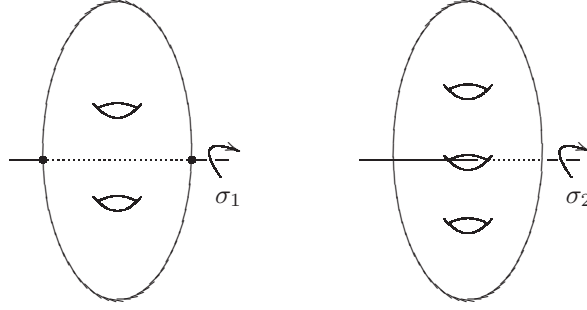
*Proof.* For part (1), use Corollaries 7.9 and 9.7, while for part (2), use Propositions 4.6 and 9.5.  $\square$

In general, though, the presence of (elliptic) pencils with multiple fibers drastically changes the nature of the  $\Omega$ -invariants.

**Proposition 9.12.** *Let  $M$  be a smooth, complex projective variety, and suppose  $M$  admits an orbifold fibration with multiple fibers and base genus  $g = 1$ . Then  $\Omega_2^1(M)$  is not an open subset of  $\text{Gr}_2(H^1(M, \mathbb{Q}))$ .*

*Proof.* By hypothesis, there is an orbifold fibration,  $f: M \rightarrow (\Sigma_1, \mathbf{m})$ , with  $|\mathbf{m}| \geq 2$ . Thus,  $\mathcal{W}^1(M)$  contains a component of the form  $\rho T$ , where  $T = f^*(H^1(\Sigma_1, \mathbb{C}^\times))$  is a 2-dimensional subtorus, and  $\rho$  is a finite-order character (modulo  $T$ ),  $\rho \notin T$ , yet  $T$  itself is not a component of  $\mathcal{W}^1(M)$ .

By Theorem 9.3, all positive-dimensional components of  $\mathcal{W}^1(M)$  are torsion-translated subtori, of the form  $\rho_\alpha T_\alpha$ . If  $T_\alpha = T$ , then the component  $\rho_\alpha T_\alpha$  must arise from an elliptic pencil; thus,  $\rho_\alpha \notin T_\alpha$ . On the other hand, if  $T_\alpha \neq T$ , then Theorem 9.8 insures that  $\tau_1(T_\alpha) \cap \tau_1(T) = \{0\}$ . Thus, all conditions of Theorem 8.5 are satisfied, and the desired conclusion follows.  $\square$


 FIGURE 1. The complex curves  $C_1$  and  $C_2$ , with involutions  $\sigma_1$  and  $\sigma_2$ 

**9.5. The Catanese–Ciliberto–Mendes Lopes surface.** We now give a concrete example of a smooth, complex projective variety  $M$  for which one of the Dwyer–Fried sets is not open. The variety in question is a minimal surface of general type with  $p_g(M) = q(M) = 3$  and  $K_M^2 = 8$ . The construction goes back to Catanese, Ciliberto, and Mendes Lopes [14]; it was revisited by Hacon and Pardini [27], and more recently, by Akhmedov and Park [2], as well as Akbulut [1].

**Example 9.13.** Let  $C_1$  be a (smooth, complex) curve of genus 2 with an elliptic involution  $\sigma_1$  and  $C_2$  a curve of genus 3 with a free involution  $\sigma_2$  (see Figure 1). Then  $\Sigma_1 = C_1/\sigma_1$  is a curve of genus 1, and  $\Sigma_2 = C_2/\sigma_2$  is a curve of genus 2. We let  $\mathbb{Z}_2$  act freely on the product  $C_1 \times C_2$  via the involution  $\sigma_1 \times \sigma_2$ , and denote by  $M$  the quotient surface. Projection onto the first coordinate yields an orbifold fibration  $f_1: M \rightarrow \Sigma_1$  with two multiple fibers, each of multiplicity 2, while projection onto the second coordinate defines a holomorphic fibration  $f_2: M \rightarrow \Sigma_2$ :

$$\begin{array}{ccccc} C_2 & \xleftarrow{\text{pr}_2} & C_1 \times C_2 & \xrightarrow{\text{pr}_1} & C_1 \\ \downarrow / \sigma_2 & & \downarrow / \sigma_1 \times \sigma_2 & & \downarrow / \sigma_1 \\ \Sigma_2 & \xleftarrow{f_2} & M & \xrightarrow{f_1} & \Sigma_1 \end{array}$$

Using the procedure described in [5], we find that the fundamental group of  $M$  has presentation with generators  $x_1, \dots, x_6$  and relators

$$\begin{aligned} & [x_3^2, x_1], \quad [x_3^2, x_2], \quad [x_2, x_1][x_2^{x_3}, x_1^{x_3}], \quad [x_3, x_4][x_5, x_6], \\ & [x_1, x_4], \quad [x_2, x_4], \quad [x_1, x_5], \quad [x_2, x_5], \quad [x_1, x_6], \quad [x_2, x_6], \\ & [x_1^{x_3}, x_4], \quad [x_2^{x_3}, x_4], \quad [x_1^{x_3}, x_5], \quad [x_2^{x_3}, x_5], \quad [x_1^{x_3}, x_6], \quad [x_2^{x_3}, x_6], \end{aligned}$$

where  $z^w := w^{-1}zw$ . Identify the character group  $\text{Hom}(\pi_1(M), \mathbb{C}^\times)$  with  $(\mathbb{C}^\times)^6$ . A straightforward computation shows that the characteristic variety  $\mathcal{V}^1(M)$  has two components, corresponding to the above two pencils. The first component is the subtorus  $T_1 = \exp(L_1 \otimes \mathbb{C})$ , with  $L_1 = \{x \in \mathbb{Q}^6 \mid x_1 = x_2 = 0\}$ , while the second component is the translated subtorus  $\rho T_2$ , where  $\rho = (1, 1, -1, 1, 1, 1)$  and  $T_2 = \exp(L_2 \otimes \mathbb{C})$ , with  $L_2 = \{x \in \mathbb{Q}^6 \mid x_3 = x_4 = x_5 = x_6 = 0\}$ .

By Theorem 9.9, we have that  $\Omega_2^1(M) = \text{Gr}_2(\mathbb{Q}^6) \setminus (\sigma_2(L_1) \cup \sigma_2(L_2, \rho))$ . Clearly, the plane  $L_2$  belongs to  $\Omega_2^1(M)$ . By Proposition 9.12, the set  $\Omega_2^1(M)$  is not open, not even in the usual topology on the Grassmannian.



**Remark 9.14.** In [27], Hacon and Pardini detect the component  $T_1$  of the characteristic variety  $\mathcal{V}^1(M)$  by means of the infinitesimal description of this variety, due to Green and Lazarsfeld [25]. Although their method does not detect the translated component  $\rho T_2$ , it does distinguish topologically the surface  $M$  from the symmetric product considered in Example 9.6.

## 10. QUASI-KÄHLER MANIFOLDS

In this section, we consider the more general setting of quasi-Kähler manifolds, which includes smooth, quasi-projective varieties such as complements of plane curves or hyperplane arrangements.

**10.1. Non-compact 2-orbifolds.** Let  $\Sigma_{g,s} = \Sigma_g \setminus \{p_1, \dots, p_s\}$  be a Riemann surface of genus  $g \geq 0$  with  $s$  points removed ( $s \geq 1$ ). Then  $\Sigma_{g,s}$  is a connected, smooth, quasi-projective variety, with  $\pi_1(\Sigma_{g,s}) = F_n$ , where  $n = b_1(\Sigma_{g,s}) = 2g + s - 1$ . Thus,  $\mathcal{V}^1(\Sigma_{g,s}) = H^1(\Sigma_{g,s}, \mathbb{C}^\times)$ , unless  $g = 0$  and  $s \leq 2$ .

More generally, let us consider as in [17, 4] a non-compact 2-orbifold, i.e., a manifold  $\Sigma_{g,s}$  with marked points  $q_1, \dots, q_t$  and weight vector  $\mathbf{m} = (m_1, \dots, m_t)$ . The orbifold fundamental group  $\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,s}, \mathbf{m})$  associated to these data is the free product

$$(54) \quad \Gamma = F_n * \mathbb{Z}_{m_1} * \dots * \mathbb{Z}_{m_t}.$$

Thus,  $\Gamma_{\text{ab}} = \mathbb{Z}^n \oplus A$ , where  $A = \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_t}$ . Identify  $\widehat{\Gamma} = (\mathbb{C}^\times)^n \times \widehat{A}$ . A Fox calculus computation as in [4, Proposition 3.10] shows that

$$(55) \quad \mathcal{V}^1(\Gamma) = \begin{cases} \widehat{\Gamma} & \text{if } n \geq 2, \\ (\widehat{\Gamma} \setminus \widehat{\Gamma}^\circ) \cup \{1\} & \text{if } n = 1 \text{ and } t > 0, \\ \{1\} & \text{if } n = 1 \text{ and } t = 0. \end{cases}$$

In the above, note that  $n = 1$  if and only if  $g = 0$  and  $s = 2$ , i.e.,  $\Sigma_{g,s} = \mathbb{C}^\times$ .

Proceeding as in the proof of Proposition 9.1, we obtain the following result (see also [17, 4]).

**Proposition 10.1.** *Let  $G$  be a finitely generated group, and suppose there is an epimorphism  $\varphi: G \twoheadrightarrow \Gamma$ , where  $\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,s}, (m_1, \dots, m_t))$ , with  $g \geq 0$  and  $s \geq 1$ . Set  $A = \text{Tors}(\Gamma_{\text{ab}})$  and  $n = 2g + s - 1$ , and suppose further that  $n \geq 2$ , or  $n = 1$  and  $t > 0$ . Then  $\mathcal{V}^1(G)$  contains all the translated subtori in  $\widehat{G}$  of the form  $\rho \cdot T$ , where  $T$  is obtained by pulling back the torus  $\widehat{F}_n = (\mathbb{C}^\times)^n$  along the homomorphism  $G \xrightarrow{\varphi} \Gamma \twoheadrightarrow F_n$ , and  $\rho$  belongs to  $\widehat{\varphi(A)}$  in the first case, and  $\widehat{\varphi(A)} \setminus \{1\}$  in the second case.*

A particular case is worth singling out.

**Corollary 10.2.** *Suppose  $G$  has a factor group of the form  $\mathbb{Z} * \mathbb{Z}_m$ , with  $m \geq 2$ . There is then a 1-dimensional algebraic subtorus  $T$  and a torsion character  $\rho \in \widehat{G} \setminus T$  such that  $\mathcal{V}^1(G)$  contains the translated tori  $\rho T, \dots, \rho^{m-1}T$ .*

**Example 10.3.** Let  $G = \langle x_1, x_2 \mid x_1 x_2^2 = x_2^2 x_1 \rangle$ , and identify  $\widehat{G} = (\mathbb{C}^\times)^2$ , with coordinates  $t_1$  and  $t_2$ . Taking the quotient of  $G$  by the normal subgroup generated by  $x_2^2$ , we obtain an epimorphism  $\varphi: G \twoheadrightarrow \mathbb{Z} * \mathbb{Z}_2$ . The morphism induced by  $\varphi$  on character groups sends  $\widehat{\mathbb{Z}}$  to  $T = \{t_2 = 1\}$ , and  $\widehat{\mathbb{Z}_2} \setminus \{1\}$  to  $\rho = (1, -1)$ . By the above proposition,  $\mathcal{V}^1(G)$  contains the translated torus  $\rho T = \{t_2 = -1\}$ . Direct computation with Fox derivatives shows that, in fact,  $\mathcal{V}^1(G) = \{1\} \cup \rho T$ .

**10.2. Arapura's theorem.** A smooth, connected manifold  $X$  is said to be a *quasi-Kähler manifold* if there is a compact Kähler manifold  $\overline{X}$  and a normal-crossings divisor  $D$  such that  $X = \overline{X} \setminus D$ . The most familiar examples of quasi-Kähler manifolds are smooth, quasi-projective complex varieties, such as complements of projective hypersurfaces, or configuration spaces of smooth projective varieties. As is well-known, every quasi-projective variety has the homotopy type of a finite CW-complex.

The basic structure of the characteristic varieties of quasi-Kähler manifolds was described by D. Arapura.

**Theorem 10.4** ([3]). *Let  $X = \overline{X} \setminus D$  be a quasi-Kähler manifold. If either  $D = \emptyset$  or  $b_1(\overline{X}) = 0$ , then each characteristic variety  $\mathcal{V}_d^i(X)$  is a finite union of unitary translates of algebraic subtori of  $\widehat{\pi_1(X)}$ .*

In the above, each component of  $\mathcal{V}_d^i(X)$  is of the form  $\rho \cdot \widehat{f_\#(\pi_1(T))}$ , for some character  $\rho: \pi_1(X) \rightarrow S^1$ , and some holomorphic map  $f: M \rightarrow T$  to a complex Lie group  $T$  which decomposes as a product of factors of the form  $\mathbb{C}^\times$  and  $S^1 \times S^1$ .

When  $X$  is quasi-projective and  $i = 1$ , a more precise structure theorem holds. Since the compact case was already treated in §9, we will focus for the rest of this section on the non-compact case.

Let  $(\Sigma_{g,s}, \mathbf{m})$  be a Riemann surface of genus  $g \geq 0$ , with  $s \geq 0$  points removed (so that  $\Sigma_{g,0} = \Sigma_g$ ), and with marked points  $(q_1, m_1), \dots, (q_t, m_t)$ . A surjective, holomorphic map  $f: X \rightarrow \Sigma_{g,s}$  is called an *orbifold fibration* (or, a pencil) if the generic fiber is connected, the multiplicity of the fiber over each  $q_i$  equals  $m_i$ , and  $f$  has an extension to the respective compactifications,  $\bar{f}: \overline{X} \rightarrow \Sigma_g$ , which is also a surjective, holomorphic map with connected generic fibers.

An orbifold fibration as above induces an epimorphism  $f_\#: \pi_1(X) \twoheadrightarrow \Gamma$ , where  $\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,s}, \mathbf{m})$  is the orbifold fundamental group described in (45) if  $s = 0$  and (54) if  $s > 0$ . By Propositions 9.1 and 10.1, the induced morphism,  $\widehat{f_\#}: \widehat{\Gamma} \hookrightarrow \widehat{\pi_1(X)}$ , sends  $\mathcal{V}^1(\Gamma)$  to a union of (possibly torsion-translated) subtori inside  $\mathcal{V}^1(X)$ .

**Theorem 10.5** ([3, 17, 4]). *Let  $X$  be a smooth, quasi-projective variety. Then*

$$(56) \quad \mathcal{V}^1(X) = \bigcup_{\alpha} \widehat{(f_\alpha)_\#}(\mathcal{V}^1(\pi_1^{\text{orb}}(\Sigma_{g_\alpha, s_\alpha}, \mathbf{m}_\alpha))) \cup Z,$$

where  $Z$  is a finite set of torsion characters, and the union runs over a finite set of orbifold fibrations  $f_\alpha: X \rightarrow (\Sigma_{g_\alpha, s_\alpha}, \mathbf{m}_\alpha)$ .

In particular, every positive-dimensional component of  $\mathcal{V}^1(X)$  is of the form  $\rho \cdot T$ , where  $T$  is an algebraic subtorus in  $H^1(X, \mathbb{C}^\times)$ , and  $\rho$  is of finite order (modulo  $T$ ). If this component arises from an orbifold fibration with base  $\Sigma_{g,s}$ , then  $T$  has dimension  $n := b_1(\Sigma_{g,s})$ . Moreover, if  $\rho \in T$ , then  $n = 2g \geq 4$  or  $n = 2g + s - 1 \geq 2$ , according to whether  $s = 0$  or not. On the other hand, if  $\rho \notin T$ , then the direction torus  $T$  is *not* a component of  $\mathcal{V}^1(X)$  precisely when  $n = 2$  (if  $s = 0$ ) or  $n = 1$  (if  $s > 0$ ).

More information on the pencils  $f_\alpha$  occurring in (56) can be found in [20, Proposition 7.2]. In particular, if  $b_1(\overline{X}) = 0$ , then all those pencils have base genus  $g_\alpha = 0$ .

Next, we derive some immediate corollaries of the above theorem.

**Corollary 10.6.** *Let  $X$  be a smooth, quasi-projective variety, and suppose  $X$  admits no orbifold fibrations with multiple fibers. Then*

$$(57) \quad \mathcal{V}^1(X) = \bigcup_{\alpha} f_\alpha^*(H^1(\Sigma_{g_\alpha, s_\alpha}, \mathbb{C}^\times)) \cup Z,$$

where  $Z$  is a finite set of torsion characters, the union runs over the set of orbifold fibrations  $f_\alpha: X \rightarrow \Sigma_{g_\alpha, s_\alpha}$  with  $2g_\alpha + s_\alpha \geq 3$ , and  $f_\alpha^*: H^1(\Sigma_{g_\alpha, s_\alpha}, \mathbb{C}^\times) \hookrightarrow H^1(X, \mathbb{C}^\times)$  is the induced homomorphism in cohomology.

**Corollary 10.7.** *Let  $X$  be a smooth, quasi-projective variety. Then*

$$(58) \quad \tau_1(\mathcal{W}^1(X)) = \bigcup_{\alpha} f_\alpha^*(H^1(\Sigma_{g_\alpha, s_\alpha}, \mathbb{C})),$$

where the union runs over all orbifold fibrations  $f_\alpha: M \rightarrow \Sigma_{g_\alpha, s_\alpha}$  with  $2g_\alpha + s_\alpha \geq 3$ , and  $f_\alpha^*: H^1(\Sigma_{g_\alpha, s_\alpha}, \mathbb{C}) \hookrightarrow H^1(X, \mathbb{C})$  is the induced homomorphism in cohomology.

The exact analogue of Theorem 9.8 also holds for quasi-projective varieties.

**10.3. Dwyer–Fried invariants.** Much as in the compact case, the above structural results inform on the  $\Omega$ -sets of (non-compact) quasi-Kähler manifolds.

For instance, let  $X = \overline{X} \setminus D$ , where  $\overline{X}$  is a compact Kähler manifold,  $D$  is a normal-crossings divisor, and  $b_1(\overline{X}) = 0$  if  $D \neq \emptyset$ . Furthermore, suppose  $\mathcal{W}^i(X)$  contains no positive-dimensional translated subtori. Then Corollary 7.7 and Theorem 10.4 insure that the inclusion  $\Omega_r^i(X) \subseteq \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X)))^0$  holds as equality, for all  $r \geq 1$ . In general, though, the inclusion can very well be strict, as the next example illustrates.

**Example 10.8.** Let  $\mathcal{C}$  be the affine plane curve with equation  $x^p - y^q = 0$ , where  $p$  and  $q$  are coprime positive integers. Its complement,  $Y = \mathbb{C}^2 \setminus \mathcal{C}$ , is homotopy equivalent to the complement in  $S^3$  of a  $(p, q)$ -torus knot  $K$ , whose Alexander polynomial is  $\Delta_K = (t^{pq} - 1)(t - 1)(t^p - 1)^{-1}(t^q - 1)^{-1}$ .

Now let  $X = Y \times (\mathbb{CP}^1 \setminus \{n \text{ points}\})$ , for some  $n \geq 3$ . Clearly,  $X$  is a smooth, quasi-projective variety, with  $H_1(X, \mathbb{Z}) = \mathbb{Z}^n$ . Identify the character group of  $\pi_1(X)$  with  $(\mathbb{C}^\times)^n$ . Using Corollary 2.10, as well as Examples 2.6 and 2.8, we see that

$$\mathcal{W}^2(X) = \{1\} \cup \bigcup_{\zeta: \Delta_K(\zeta)=0} \{\zeta\} \times (\mathbb{C}^\times)^{n-1}.$$

Since  $\Delta_K(1) \neq 0$ , we have that  $\tau_1(\mathcal{W}^2(X)) = \{0\}$ . On the other hand, Proposition 8.6 gives that  $\Omega_r^2(X) = \text{Gr}_r(\{0\} \times \mathbb{Q}^{n-1})$  if  $1 < r < n$  and  $\Omega_r^2(X) = \emptyset$  if  $r = n$ . In the first case,  $\Omega_r^2(X)$  is *not* an open subset of  $\text{Gr}_r(\mathbb{Q}^n)$ , and in either case,  $\Omega_r^2(X)$  is strictly included in  $\sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^2(X)))^0 = \text{Gr}_r(\mathbb{Q}^n)$ .

In degree  $i = 1$ , we can say more. As before, let  $X$  be a smooth (connected, non-compact) quasi-projective variety. Given an orbifold fibration  $f: X \rightarrow (\Sigma_{g, s}, \mathbf{m})$ , set  $L = f^*(H^1(\Sigma_{g, s}, \mathbb{Q}))$  and  $A = \text{Tors}(\pi_1^{\text{orb}}(\Sigma_{g, s}, \mathbf{m}))$ , and denote by  $\varphi: \pi_1(X) \twoheadrightarrow \pi_1^{\text{orb}}(\Sigma_{g, s}, \mathbf{m})$  the homomorphism induced by  $f$ .

**Proposition 10.9.** *Let  $X$  be a smooth, quasi-projective variety. For all  $r \geq 1$ ,*

$$(59) \quad \Omega_r^1(X) = \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \bigcup_{\alpha} \bigcup_{\rho} \sigma_r(L_\alpha, \rho),$$

where the first union runs over the set of equivalence classes of orbifold fibrations  $f_\alpha: X \rightarrow (\Sigma_{g_\alpha, s_\alpha}, \mathbf{m}_\alpha)$  with either

- (1)  $2g_\alpha + s_\alpha \geq 3$ , or
- (2)  $g_\alpha = 1$ ,  $s_\alpha = 0$  and  $|\mathbf{m}_\alpha| \geq 2$ , or
- (3)  $g_\alpha = 0$ ,  $s_\alpha = 2$  and  $|\mathbf{m}_\alpha| \geq 1$ ,

while the second union runs over all  $\rho \in \hat{\varphi}_\alpha(\hat{A}_\alpha)$  in case (1) or  $\rho \in \hat{\varphi}_\alpha(\hat{A}_\alpha) \setminus \{1\}$  in cases (2) and (3).

*Proof.* In view of Propositions 9.1 and 10.1, as well as Theorem 10.5, the conclusion follows from Theorem 8.4.  $\square$

**Proposition 10.10.** *Let  $X$  be a smooth, quasi-projective variety. Then, for all  $r \geq 1$ ,*

$$(60) \quad \Omega_r^1(X) \subseteq \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \bigcup_{\alpha} \sigma_r(f_{\alpha}^*(H^1(\Sigma_{g_{\alpha}, s_{\alpha}}, \mathbb{Q}))),$$

where the union runs over the set of orbifold fibrations  $f_{\alpha}: X \rightarrow \Sigma_{g_{\alpha}, s_{\alpha}}$  with  $2g_{\alpha} + s_{\alpha} \geq 3$ . Moreover, if  $r = 1$ , or there are no orbifold fibrations with multiple fibers, then (60) holds as equality.

*Proof.* Formula (60) is a consequence of Theorem 7.4 and Corollary 10.7. The last two claims follow from Theorem 6.4 and Corollaries 7.7 and 10.6, respectively.  $\square$

Using Corollary 7.9, we obtain the following immediate consequence.

**Corollary 10.11.** *Suppose there is an orbifold fibration  $f: X \rightarrow \Sigma_{g,s}$  with  $2g + s \geq 3$ . Then  $\Omega_r^1(X) = \emptyset$ , for all  $r > b_1(X) - b_1(\Sigma_{g,s})$ .*

On the other hand, as the next result shows, if the characteristic variety  $\mathcal{W}^1(X)$  contains positive-dimensional translated components, its exponential tangent cone may fail to determine the Dwyer–Fried sets  $\Omega_r^1(X)$  with  $r > 1$ .

**Proposition 10.12.** *Let  $X$  be a smooth, quasi-projective variety. Suppose that  $\mathcal{W}^1(X)$  has a 1-dimensional component not passing through 1, and  $\tau_1(\mathcal{W}^1(X))$  has codimension greater than 1. Then  $\Omega_2^1(X)$  is strictly contained in  $\sigma_2(\tau_1^{\mathbb{Q}}(\mathcal{W}^1(X))^{\circ}$ .*

A proof of this result is given in [45, Theorem 10.11], in the particular case when the quasi-projective variety  $X$  is 1-formal. The argument given there extends without any essential modifications to this more general setting, upon replacing the resonance variety  $\mathcal{R}^1(X, \mathbb{C})$  by the exponential tangent cone  $\tau_1(\mathcal{W}^1(X))$ .

**Corollary 10.13.** *Let  $X$  be a smooth, quasi-projective variety. Suppose there is a pencil  $X \rightarrow \mathbb{C}^{\times}$  with multiple fibers, but there is no pencil  $X \rightarrow \Sigma_{g,s}$  with  $b_1(X) - b_1(\Sigma_{g,s}) \leq 1$ . Then  $\Omega_2^1(X) \subsetneq \mathrm{Gr}_2(H^1(X, \mathbb{Q})) \setminus \sigma_2(\tau_1^{\mathbb{Q}}(\mathcal{W}^1(X)))$ .*

*Proof.* Follows from Proposition 10.12, formulas (47) and (55), and Theorem 10.5.  $\square$

## 11. HYPERPLANE ARRANGEMENTS

Among quasi-projective varieties, complements of complex hyperplane arrangements stand out as a particularly fascinating class of examples.

**11.1. Characteristic varieties of arrangements.** Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be a finite set of hyperplanes in some finite-dimensional complex vector space  $\mathbb{C}^d$ . Most of the time, we will assume the arrangement is central, i.e., all hyperplanes pass through the origin. In this case, a defining polynomial for the arrangement is the product  $Q = \prod_{i=1}^n \alpha_i$ , where  $\alpha_i: \mathbb{C}^d \rightarrow \mathbb{C}$  are linear forms with  $\ker(\alpha_i) = H_i$ .

Let  $X(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup_{i=1}^n H_i$  be the complement of the arrangement. This is a smooth, quasi-projective variety, with the homotopy type of a connected, finite CW-complex of dimension  $d$ . The cohomology ring  $H^*(X(\mathcal{A}), \mathbb{Z})$  is the quotient of the exterior algebra on classes dual to the meridians around the hyperplanes, modulo a certain ideal generated in degrees  $\geq 2$ . As shown by Orlik and Solomon in the 1980s, this ideal is determined by the intersection lattice, that is, the poset  $L(\mathcal{A})$  of all non-empty intersections of  $\mathcal{A}$ , ordered by reverse inclusion.

Using the meridian basis of  $H_1(X(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^n$ , we may identify the character group of  $\pi_1(X(\mathcal{A}))$  with the complex torus  $(\mathbb{C}^\times)^n$ . By Theorem 10.4, the characteristic varieties  $\mathcal{V}_1^i(X(\mathcal{A}))$  are unions of unitary translates of algebraic subtori in  $(\mathbb{C}^\times)^n$ . If  $\mathcal{A}$  is central, then its complement is diffeomorphic to the product of  $\mathbb{C}^\times$  with the complement in  $\mathbb{CP}^{d-1}$  of the projectivization of  $\mathcal{A}$ . From Proposition 2.9, we see that  $\mathcal{V}_1^i(X(\mathcal{A}))$  is a subvariety of the complex torus  $\{t \in (\mathbb{C}^\times)^n \mid t_1 \cdots t_n = 1\} \cong (\mathbb{C}^\times)^{n-1}$ .

We will be only interested here in the degree-one characteristic variety,  $\mathcal{V}^1(\mathcal{A}) := \mathcal{V}^1(X(\mathcal{A}))$ . By Theorem 10.5, this variety is a union of torsion-translated subtori. Results from [15] and [20] imply that the exponential tangent cone to  $\mathcal{V}^1(\mathcal{A})$  coincides with the resonance variety  $\mathcal{R}^1(X(\mathcal{A}))$ . Thus, the components of  $\mathcal{V}^1(\mathcal{A})$  passing through the origin are determined by the intersection lattice of  $\mathcal{A}$ .

If  $\mathcal{B} \subset \mathcal{A}$  is a sub-arrangement, the inclusion  $X(\mathcal{A}) \hookrightarrow X(\mathcal{B})$  induces an epimorphism  $\pi_1(X(\mathcal{A})) \twoheadrightarrow \pi_1(X(\mathcal{B}))$ . By Lemma 2.13, the resulting monomorphism between character groups restricts to an embedding  $\mathcal{V}^1(\mathcal{B}) \hookrightarrow \mathcal{V}^1(\mathcal{A})$ . Components of  $\mathcal{V}^1(\mathcal{A})$  which are not supported on any proper sub-arrangement are said to be essential.

**11.2. Pencils, multinet, and translated tori.** Let us describe in finer detail the geometry of the characteristic variety  $\mathcal{V}^1(\mathcal{A})$ . We start with a simple, yet basic example.

**Example 11.1.** Let  $\mathcal{A}$  be an arrangement of  $n \geq 3$  lines through the origin of  $\mathbb{C}^2$ , defined by the polynomial  $Q = x^n - y^n$ . Then  $X(\mathcal{A})$  is diffeomorphic to  $\mathbb{C}^\times \times \Sigma_{0,n}$ . Using Example 2.6 and Proposition 2.9, we see that  $\mathcal{V}^1(\mathcal{A}) = (\mathbb{C}^\times)^{n-1}$ . Clearly, this single component arises as pull-back along the projection map  $X(\mathcal{A}) \rightarrow \Sigma_{0,n}$ .

More generally, let  $\mathcal{A}$  be a central arrangement in  $\mathbb{C}^d$ . Since  $X(\mathcal{A})$  admits a non-singular compactification with  $b_1 = 0$ , every orbifold fibration  $f: X(\mathcal{A}) \rightarrow \Sigma_{g,s}$  must have base genus  $g = 0$ .

Taking a generic 2-section, we obtain an arrangement  $\mathcal{A}'$  of affine lines in  $\mathbb{C}^2$ , with  $\pi_1(X(\mathcal{A})) = \pi_1(X(\mathcal{A}'))$ ; therefore,  $\mathcal{V}^1(\mathcal{A}) = \mathcal{V}^1(\mathcal{A}')$ . Each intersection point of multiplicity  $s \geq 3$  in  $\mathcal{A}'$  gives rise to a “local” component of  $\mathcal{V}^1(\mathcal{A})$  of dimension  $s - 1$ . Such components arise as pull-backs along pencils  $X(\mathcal{A}) \rightarrow \Sigma_{0,s}$  similar to the ones from Example 11.1. In general, though, there are non-local components in  $\mathcal{V}^1(\mathcal{A})$ .

**Example 11.2.** Let  $\mathcal{A}$  be the braid arrangement in  $\mathbb{C}^3$ , defined by the polynomial  $Q = (x^2 - y^2)(x^2 - z^2)(y^2 - z^2)$ . The variety  $\mathcal{V}^1(\mathcal{A}) \subset (\mathbb{C}^\times)^6$  has 4 local components of dimension 2, corresponding to 4 triple points in a generic section. Additionally, the rational map  $f: \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^1$  given by  $f(x, y, z) = (x^2 - y^2, x^2 - z^2)$  restricts to a holomorphic fibration  $f: X(\mathcal{A}) \rightarrow \Sigma_{0,3}$ , where  $\Sigma_{0,3} = \mathbb{CP}^1 \setminus \{(1, 0), (0, 1), (1, 1)\}$ . This yields an essential, 2-dimensional component in  $\mathcal{V}^1(\mathcal{A})$ .

For an arbitrary central arrangement  $\mathcal{A}$  in  $\mathbb{C}^3$  with defining polynomial  $Q = \prod_{i=1}^n \alpha_i$ , all the irreducible components of  $\mathcal{V}^1(\mathcal{A})$  passing through the origin can be described in terms of “multinets” on the intersection lattice of  $\mathcal{A}$ . As shown by Falk, Pereira, and Yuzvinsky in [24, 36, 50], every such multinet determines a completely reducible curve  $\{Q^\mu = 0\}$  in  $\mathbb{CP}^2$ , where  $Q^\mu = \prod_{i=1}^n \alpha_i^{\mu_i}$ , for some vector  $\mu \in \mathbb{Z}^n$ . In turn, this curve defines a pencil  $X(\mathcal{A}) \rightarrow \Sigma_{0,s}$  for some  $s \geq 3$ , and this pencil produces an  $(s - 1)$ -dimensional component of  $\mathcal{V}^1(\mathcal{A})$ , passing through 1. Moreover, if this component is non-local, then  $s = 3$  or 4.

In general, though, the characteristic variety  $\mathcal{V}^1(\mathcal{A})$  has irreducible components not passing through the origin.

**Example 11.3.** Let  $\mathcal{A}$  be the deleted  $B_3$  arrangement, with defining polynomial  $Q = xyz(x-y)(x-z)(y-z)(x-y-z)(x-y+z)$ ; a generic plane section of  $\mathcal{A}$  is depicted in Figure 2. The characteristic variety  $\mathcal{V}^1(\mathcal{A}) \subset (\mathbb{C}^\times)^8$  was computed in [42]: it has six local components of dimension 2, one local component of dimension 3, five non-local components of dimension 2 corresponding to braid sub-arrangements, and an essential component of the form  $\rho T$ , where  $\rho = \exp(2\pi i\lambda)$ , with  $\lambda = (1/2, 0, 1/2, 1/2, 0, 1/2, 0, 0)$ , and  $T = \exp(\ell \otimes \mathbb{C})$ , with  $\ell$  the line in  $\mathbb{Q}^8$  spanned by  $\mu = (-1, 1, 0, 0, 1, -1, -2, 2)$ .

Since  $\dim T = 1$  and  $\rho$  has order 2, the component  $\rho T$  must arise by pullback along a pencil  $X(\mathcal{A}) \rightarrow (\Sigma_{0,2}, 2)$  with a single multiple fiber, of multiplicity 2. As noted in [17], this pencil is  $Q^\mu: X(\mathcal{A}) \rightarrow \mathbb{C}^\times$ , and the multiple fiber is the one over  $1 \in \mathbb{C}^\times$ .

Let us summarize the above discussion, as follows. As before, let  $\mathcal{A}$  be an arrangement of  $n$  hyperplanes.

**Theorem 11.4.** *Each irreducible component of  $\mathcal{V}^1(\mathcal{A})$  is a torsion-translated subtorus of  $(\mathbb{C}^\times)^n$ . Moreover, each positive-dimensional, non-local component is of the form  $\rho T$ , where  $\rho$  is a torsion character,  $T = f^*(H^1(\Sigma_{0,s}, \mathbb{C}^\times))$ , for some orbifold fibration  $f: X(\mathcal{A}) \rightarrow (\Sigma_{0,s}, \mathbf{m})$ , and either*

- (1)  $s = 2$ , and  $f$  has at least one multiple fiber, or
- (2)  $s = 3$  or  $4$ , and  $f$  corresponds to a multinet on  $L(\mathcal{A})$ .

**Remark 11.5.** As shown in [42], there do exist arrangements  $\mathcal{A}$  with isolated torsion points in  $\mathcal{V}^1(\mathcal{A})$ . Moreover, as noted in Example 11.3, there also exist arrangements  $\mathcal{A}$  with 1-dimensional translated tori (of type (1)) in  $\mathcal{V}^1(\mathcal{A})$ . On the other hand, we do not know whether higher-dimensional translated tori occur (these would necessarily be of type (2), and the corresponding orbifold fibrations would have multiple fibers).

Of course, the local components in  $\mathcal{V}^1(\mathcal{A})$  are determined by the intersection lattice of  $\mathcal{A}$ . Moreover, the components of type (2) corresponding to orbifold fibrations with no multiple fibers are also determined by  $L(\mathcal{A})$ . It is not known whether the remaining components (i.e., those of dimension 0, those of type (1), and those of type (2) having non-trivial translation factors) are also combinatorially determined.

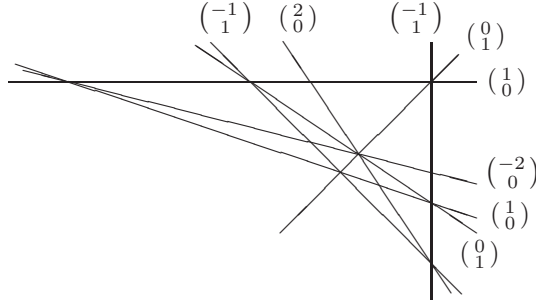
**11.3. Dwyer–Fried invariants of arrangements.** Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be a central hyperplane arrangement. A connected, regular  $\mathbb{Z}^r$ -cover of the complement  $X(\mathcal{A})$  is specified by assigning to each hyperplane  $H_i$  the  $i$ -th column of an integral  $r \times n$  matrix of rank  $r$ . In turn, such a matrix defines a point in the Grassmannian of  $r$ -planes in  $\mathbb{Q}^n$ . The Dwyer–Fried invariants of the arrangement,  $\Omega_r^i(\mathcal{A}) = \Omega_r^i(X(\mathcal{A}))$ , record the locus of points in  $\text{Gr}_r(\mathbb{Q}^n)$  for which the first  $i$  Betti numbers of the corresponding cover are finite. From Proposition 10.9, we know that

$$(61) \quad \Omega_r^1(\mathcal{A}) = \text{Gr}_r(\mathbb{Q}^n) \setminus \bigcup_{\alpha} \bigcup_{\rho} \sigma_r(L_\alpha, \rho),$$

where the first union runs over the set of orbifold fibrations  $f_\alpha: X(\mathcal{A}) \rightarrow (\Sigma_{0,s_\alpha}, \mathbf{m}_\alpha)$  with either  $s_\alpha \geq 3$ , or  $s_\alpha = 2$  and  $|\mathbf{m}_\alpha| > 0$ , while the second union runs over all characters (respectively, non-trivial characters)  $\rho$  in  $(\widehat{f_\alpha})_2(\widehat{A_\alpha})$ , and where  $L_\alpha = f_\alpha^*(H^1(\Sigma_{0,s_\alpha}, \mathbb{Q}))$  and  $A_\alpha = \text{Tors}(\pi_1^{\text{orb}}(\Sigma_{0,s_\alpha}, \mathbf{m}_\alpha))$ .

Note that the linear subspaces  $L_\alpha \subset \mathbb{Q}^n$  comprise the characteristic subspace arrangement  $\mathcal{C}_1(X(\mathcal{A}))$ . Each such subspace has dimension  $c_\alpha = s_\alpha - 1$ , with  $1 < c_\alpha < n$ . Moreover, if  $\exp(L_\alpha \otimes \mathbb{C})$  is not a local component, then  $c_\alpha = 2$  or  $3$ . By Corollary 10.11, the set  $\Omega_r^1(\mathcal{A})$  is empty for all  $r > n - c$ , where  $c = \max\{c_\alpha\}$ . In particular, if a generic 2-section of  $\mathcal{A}$  has an intersection point of multiplicity  $s \geq 3$ , then  $\Omega_{n-s+2}^1(\mathcal{A}) = \emptyset$ .



FIGURE 2. Generic section of the deleted  $B_3$  arrangement and a  $\mathbb{Z}^2$ -cover

In view of Remark 11.5, it remains an open question whether the Dwyer–Fried sets of an arrangement are combinatorially determined. Nevertheless, Proposition 10.10 yields the following combinatorial upper bound for these sets:

$$(62) \quad \Omega_r^1(\mathcal{A}) \subseteq \text{Gr}_r(\mathbb{Q}^n) \setminus \bigcup_{\alpha} \sigma_r(L_{\alpha}),$$

where the union runs over the set of orbifold fibrations  $f_{\alpha}: X(\mathcal{A}) \rightarrow \Sigma_{0,s_{\alpha}}$ ,  $s_{\alpha} \geq 3$ , corresponding to multinets on  $L(\mathcal{A})$ . Of course, if either  $r = 1$ , or there are no orbifold fibrations with multiple fibers, then the above inclusion holds as equality.

**Example 11.6.** Let  $\mathcal{A}$  be an arrangement of  $n$  planes in  $\mathbb{C}^3$ , and suppose a generic 2-section has one or two lines which contain all the intersection points of multiplicity 3 and higher. Then, as shown by Nazir and Raza in [34], the variety  $\mathcal{V}^1(\mathcal{A})$  has no translated components. Thus, (62) holds as equality in this case.

Here is an example where the above condition is not satisfied, yet equality is still attained in (62).

**Example 11.7.** If  $\mathcal{A}$  is the braid arrangement from Example 11.2, then all components of  $\mathcal{V}^1(\mathcal{A})$  pass through the origin, and  $\mathcal{C}_1(X(\mathcal{A}))$  consists of 5 planes in  $\mathbb{Q}^6$ . Thus,  $\Omega_r^1(\mathcal{A}) = \text{Gr}_r(\mathbb{Q}^6) \setminus \bigcup_{i=1}^5 \sigma_r(L_i)$  for  $r \leq 4$ , and  $\Omega_r^1(\mathcal{A}) = \emptyset$  for  $r > 4$ .

In general, though, translated tori in the characteristic variety  $\mathcal{V}^1(\mathcal{A})$  will affect the  $\Omega$ -invariants of an arrangement  $\mathcal{A}$ .

**Example 11.8.** Let  $\mathcal{A}$  be the deleted  $B_3$  arrangement from Example 11.3. We know that the characteristic subspace arrangement  $\mathcal{C} = \mathcal{C}_1(X(\mathcal{A}))$  consists of eleven 2-planes and a 3-plane in  $\mathbb{Q}^8$ . Moreover, we know that  $\mathcal{V}^1(\mathcal{A})$  contains a 1-dimensional translated torus. Therefore, Corollary 10.13 tells us that  $\Omega_2^1(\mathcal{A})$  is strictly contained in the set  $U = \text{Gr}_2(\mathbb{Q}^8) \setminus \bigcup_{L \in \mathcal{C}} \sigma_2(L)$ .

In the notation from Example 11.3, an explicit element in  $U \setminus \Omega_2^1(\mathcal{A})$  is the plane  $P$  spanned by the vectors  $\mu$  and  $2\lambda$ . The corresponding  $\mathbb{Z}^2$ -cover of  $X(\mathcal{A})$  is obtained by sending the meridians to the vectors indicated in Figure 2.

## 12. FINITENESS PROPERTIES OF GROUPS

In this final section, we discuss some alternate ways to measure the finiteness properties of a group. It turns out that the  $\Omega$ -invariants give subtle information about those properties, too.



**12.1. Properties  $F_k$  and  $FP_k$ .** Perhaps the most basic finiteness property for discrete groups is the geometric finiteness condition introduced by C.T.C. Wall in [49]. A group  $G$  is said to satisfy property  $F_k$ , for some  $k \geq 1$ , if  $G$  admits a classifying space  $K(G, 1)$  with finite  $k$ -skeleton. Evidently, the  $F_1$  property is equivalent to  $G$  being finitely generated, while the  $F_2$  property is equivalent to  $G$  being finitely presentable.

These notions have an algebraic analogue, introduced by J.P. Serre [38] and R. Bieri [8]. A group  $G$  is said to satisfy property  $FP_k$ , for some  $k \geq 1$ , if the trivial  $\mathbb{Z}G$ -module  $\mathbb{Z}$  admits a partial resolution,  $P_k \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$ , with all  $P_i$  finitely generated, projective  $\mathbb{Z}G$ -modules.

Clearly, if  $G$  is of type  $F_k$ , then it is also of type  $FP_k$ . For  $k = 1$ , the two conditions are equivalent. Furthermore, if  $G$  is finitely presented and of type  $FP_k$ , then  $G$  is of type  $F_k$ . For each  $k \geq 2$ , Bestvina and Brady produced in [7] examples of groups of type  $FP_k$ , but not of type  $F_k$ . Finally, if  $G$  is of type  $FP_k$ , then  $H_i(G, \mathbb{Z})$  is finitely generated, for all  $i \leq k$ , and thus  $b_i(G) < \infty$ , for all  $i \leq k$ . In general, though, none of these implications can be reversed (see Example 3.8 for one instance of this phenomenon).

**12.2. Dwyer–Fried invariants and finiteness properties.** We now present a concrete way in which the  $\Omega$ -sets can be used to inform on the aforementioned finiteness properties of groups. The idea goes back to Stallings’ seminal paper [40]; we refer to Example 12.3 and the proof of Theorem 12.5 for some concrete applications.

**Theorem 12.1.** *Let  $G$  be a finitely generated group, and let  $\nu: G \twoheadrightarrow \mathbb{Z}^r$  be an epimorphism, with kernel  $\Gamma$ . Suppose that  $\Omega_r^k(G) = \emptyset$  and  $\Gamma$  is of type  $F_{k-1}$ . Then  $b_k(\Gamma) = \infty$ . Consequently,  $H_k(\Gamma, \mathbb{Z})$  is not finitely generated, and thus  $\Gamma$  is not of type  $FP_k$ .*

*Proof.* Set  $X = K(G, 1)$ ; then  $X^\nu = K(\Gamma, 1)$ . Since  $\Gamma$  is of type  $F_{k-1}$ , we have  $b_i(X^\nu) < \infty$  for  $i \leq k-1$ . On the other hand, since  $\Omega_r^k(X) = \emptyset$ , we must have  $b_k(X^\nu) = \infty$ . The conclusions follow.  $\square$

**Corollary 12.2.** *Let  $G$  be a finitely generated group, and suppose that  $\Omega_1^3(G) = \emptyset$ . Let  $\nu: G \twoheadrightarrow \mathbb{Z}$  be an epimorphism. If the group  $\Gamma = \ker(\nu)$  is finitely presented, then  $H_3(\Gamma, \mathbb{Z})$  is not finitely generated.*

**Example 12.3.** Let  $Y_2 = S^1 \vee S^1$  and  $X = Y_2 \times Y_2 \times Y_2$ . Clearly,  $X$  is a classifying space for the group  $G = F_2 \times F_2 \times F_2$ . Let  $\nu: G \twoheadrightarrow \mathbb{Z}$  be the homomorphism taking each standard generator to 1. In [40], Stallings found an explicit finite presentation for the group  $\Gamma = \ker(\nu)$ :

$$(63) \quad \Gamma = \langle a, b, c, x, y \mid [x, a], [y, a], [x, b], [y, b], [a^{-1}x, c], [a^{-1}y, c], [b^{-1}a, c] \rangle.$$

He then proceeded to show, via a Mayer–Vietoris argument, that  $H_3(\Gamma, \mathbb{Z})$  is not finitely generated. This last assertion can be readily explained from our point of view. Indeed, by formula (27), we have that  $\Omega_1^3(X) = \emptyset$ . The desired conclusion then follows from Corollary 12.2.

**Remark 12.4.** It turns out that the Stallings group is a (non-central) arrangement group. This fact was observed by the author during a conversation with Daniel Matei in May 2004, and announced in a talk given at MSRI in August 2004 [43]. Since the explicit computation has not appeared in print, let us record it here.

Consider the arrangement of 5 lines in  $\mathbb{C}^2$  defined by the vanishing of the polynomial  $Q = zw(w+1)(z-1)(2z+w)$ . Using the braid monodromy generators recorded in [41,

Example 10.2], we obtain the following presentation for the fundamental group of the complement of this arrangement:

$$(64) \quad G = \left\langle x_1, \dots, x_5 \left| \begin{array}{l} [x_1, x_2], [x_1, x_4], [x_2, x_4], [x_3, x_4], [x_2, x_5], \\ x_1 x_3 x_5 = x_5 x_1 x_3 = x_3 x_5 x_1 \end{array} \right. \right\rangle,$$

with generators corresponding to the lines, and relators corresponding to the intersection points. An isomorphism  $\Gamma \xrightarrow{\cong} G$  is given by  $a \mapsto x_1 x_3 x_4 x_5$ ,  $b \mapsto x_1 x_3 x_5$ ,  $c \mapsto x_2$ ,  $x \mapsto x_1 x_3$ ,  $y \mapsto x_3$ .

**12.3. Kollár's question.** Two groups,  $G_1$  and  $G_2$ , are said to be *commensurable up to finite kernels* if they can be connected by a zig-zag of groups and homomorphisms, with all arrows of finite kernel and cofinite image. In [30], J. Kollár asked the following question: *Given a smooth, projective variety  $M$ , is the fundamental group  $\Gamma = \pi_1(M)$  commensurable, up to finite kernels, with another group,  $\pi$ , admitting a  $K(\pi, 1)$  which is a quasi-projective variety?* In [19], we answered this question, as follows.

**Theorem 12.5** ([19]). *For each  $k \geq 3$ , there is a smooth, irreducible, complex projective variety  $M$  of complex dimension  $k - 1$ , such that the group  $\Gamma = \pi_1(M)$  is of type  $F_{k-1}$ , but not of type  $FP_k$ .*

Using some classical results of R. Bieri, it is easy to see that such a group  $\Gamma$  is not commensurable, up to finite kernels, to any group of type  $FP_k$ , and thus, to any group  $\pi$  admitting a  $K(\pi, 1)$  which is a quasi-projective variety. As noted in [23], the manifolds constructed in Theorem 12.5 cannot have a simply-connected holomorphic embedding into a compactifiable complex manifold with contractible universal covering space.

**12.4. Branched covers and generic fibers.** We now sketch the proof of Theorem 12.5, in a streamlined version afforded by the technology developed here. The starting point is a classical branched covering construction.

Let  $C$  be a bielliptic curve of genus  $g \geq 2$ . Then  $C$  supports an involution  $\sigma$ , such that the quotient  $C/\sigma$  is an elliptic curve  $E$ . (The case  $g = 2$  is depicted on the left side of Figure 1.) The projection map,  $f: C \rightarrow E$ , is a surjective holomorphic map, and can be viewed as a 2-fold branched cover, with branch set  $B \subset E$  consisting of  $2g - 2$  points. The corresponding unramified 2-fold cover is classified by a homomorphism  $\varphi: \pi_1(E \setminus B) \rightarrow \mathbb{Z}_2$  taking each standard loop around a branch point to 1.

Now fix an integer  $k \geq 3$ , and set  $X = C^{\times k}$ . The group law of the elliptic curve extends by associativity to a map  $s_k: E^{\times k} \rightarrow E$ . Composing this map with the product map  $f^{\times k}: C^{\times k} \rightarrow E^{\times k}$ , we obtain a surjective holomorphic map,  $h = s_k \circ f: X \rightarrow E$ .

**Lemma 12.6** ([19]). *Let  $M$  be the generic fiber of  $h$ . Then  $M$  is a smooth, complex projective variety of dimension  $k - 1$ . Moreover,*

- (1)  $M$  is connected.
- (2)  $\pi_1(M) = \ker(h_{\sharp}: \pi_1(X) \rightarrow \pi_1(E))$ .
- (3)  $\pi_2(M) = \dots = \pi_{k-2}(M) = 0$ .

Assuming this lemma, it is now an easy matter to finish the proof of the theorem.

*Proof of Theorem 12.5.* Set  $G = \pi_1(X)$  and  $\Gamma = \pi_1(M)$ . Identify  $\pi_1(E) = \mathbb{Z}^2$ , and write  $\nu = h_{\sharp}$ . From Lemma 12.6, parts (1) and (2), we obtain a short exact sequence,  $1 \rightarrow \Gamma \rightarrow G \xrightarrow{\nu} \mathbb{Z}^2 \rightarrow 1$ . Since  $X$  is a  $k$ -fold direct product of surfaces of genus  $g \geq 2$ , the space  $X$  is a  $K(G, 1)$ . Moreover, formula (26) shows that  $\Omega_2^k(G) = \emptyset$ .

In view of Lemma 12.6, part (3), a classifying space  $K(\Gamma, 1)$  can be obtained from  $M$  by attaching cells of dimension  $k$  and higher. Consequently, the group  $\Gamma$  is of type  $F_{k-1}$ . Finally, Theorem 12.1 shows that  $\Gamma$  is not of type  $FP_k$ .  $\square$

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